Forcing with Urelements

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Preliminaries

The iterative conception of set

$V_0 = \emptyset$;

$V_{\alpha+1} = P(V_\alpha)$;

$V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$, where γ is a limit;

$V = \bigcup_{\alpha<Ord} V_\alpha$.

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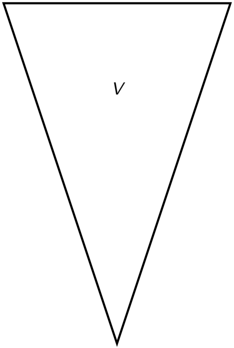

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ZFC is ZF + the Axiom of Choice (i.e., every set is well-ordered).

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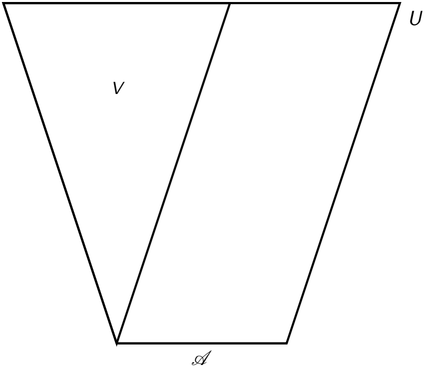
Let $\mathscr{A}$ be the class of urelements (not necessarily a set). Then we have the whole universe U as

$$U = \bigcup_{A \subseteq \mathscr{A}} V(A).$$

The universe with urelements



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They are philosophically interesting.

- Set theory with urelements is a more general ontological framework;
- different philosophical conceptions of set and the nature of many set-theoretic axiom can be better understood with urelements.

ZFU_R

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$\mathrm{ZFU_R} = \mathrm{ZU} + \text{Replacement}$.
$\mathrm{ZFCU_R} = \mathrm{ZFU_R} + \mathrm{AC}$.
$\mathrm{ZF} = \mathrm{ZFU_R} + \forall x \neg \mathscr{A}(x)$.
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ZFC = ZF + AC.

Note. The subscript R means stronger axioms are not included. It is folklore that $\mathrm{ZFCU_R}$ cannot prove the Collection Principle

$$(\text{Collection})\forall x \in w \exists y \varphi(x,y,u) \to \exists v \forall x \in w \exists y \in v\ \varphi(x,y,u).$$

Interpreting U in V

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Definition (Barwise?)

Let V be a model of ZF and X be a class of V. In V, define by recursion

$$V[\![X]\!] = (\{0\} \times X) \cup \{\bar{x} \in V : \exists x(\bar{x} = \langle 1, x\rangle \wedge x \subseteq V[\![X]\!])\}.$$

For every $\bar{x}, \bar{y} \in V[\![X]\!]$,

$\bar{x} \mathrel{\bar{\in}} \bar{y}$ iff $\exists y(\bar{y} = \langle 1, y\rangle \wedge \bar{x} \in y)$;

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Theorem

Let V be a model of ZF and X be a class of V. Then

- $V[\![X]\!] \models \mathrm{ZFU}_R$ + *Collection;*
- $V[\![X]\!] \models AC$ *iff* $V \models AC$.

Interpreting U in V

Corollary.

The following theories are mutually interpretable.

- ZF.
- $\mathrm{ZFCU_R}$ + Collection + $\mathscr{A} \sim \omega$.
- $\mathrm{ZFCU_R}$ + Collection + $\mathscr{A} \sim \omega_1$.
- $\mathrm{ZFCU_R}$ + Collection + "for every cardinal κ, there is a set of κ-many urelements".

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Proof.

$V[\![\omega_1]\!]$ has ω_1-many urelements, and $V[\![Ord]\!]$ has unboundedly many. $\square$

A Hierarchy of Axioms

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Partial reflection: any true statement is true in some transitive set containing the parameters.

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First-order reflection is often seen as a consequence of the iterative conception of set.

Dependent choice

For every infinite cardinal κ, the DC_κ-scheme is a class version of the axiom DC_κ.

> (DC_κ-scheme) If for every x there is some y such that $\varphi(x, y, u)$, then there is a κ-sequence f such that $\varphi(f \restriction \alpha, f(\alpha), u)$ for every $\alpha < \kappa$.

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Fact

- $\mathrm{ZU} + \mathrm{RP} \vdash$ Collection;
- $\mathrm{ZFU_R} + \mathscr{A}$ is a set $\vdash \mathrm{RP}$;
- $\mathrm{ZFCU_R} + \mathscr{A}$ is a set $\vdash \mathrm{DC}_{Ord}$.

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Theorem (Schlutzenberg)

$\mathrm{ZFCU}_R + \textit{Collection} \vdash \textit{DC}_\omega\textit{-Scheme}.$

Urelement axioms

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Definition

A set x is **realized** if there is a set of urelements equinumerous with x.
For any sets of urelements $A, B \subseteq \mathscr{A}$, B is a **tail** of A, if B is disjoint from A and every $C \subseteq \mathscr{A}$ disjoint from A injects into B.

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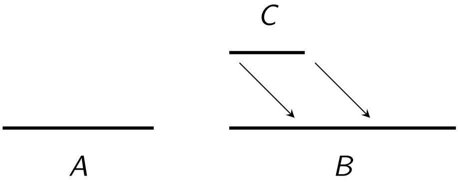

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The hierarchy in $\mathrm{ZFCU_R}$

Theorem

Over ZFCU_R, the following implication diagram holds and is complete.

$$\begin{tikzcd}
\text{Plenitude} \arrow[d] \arrow[dr] & & \mathscr{A} \text{ is a set} \arrow[dl] \arrow[d] & & \\
\text{Closure} \wedge \text{Duplication} \arrow[d] \arrow[drr] & \mathrm{DC}_{Ord} \arrow[d, dotted] \arrow[dr] & \text{Tail} \arrow[d] & & \\
\text{Duplication} & \mathrm{DC}_{\kappa}\text{-scheme} \arrow[d, dotted] & \text{Collection} \arrow[r] \arrow[d] \arrow[dr] & \mathrm{RP} \arrow[r] & \mathrm{RP}^{-} \arrow[ll, bend right] \\
& \mathrm{DC}_{\omega_1}\text{-scheme} \arrow[r] & \mathrm{DC}_{\omega}\text{-scheme} & \text{Closure} &
\end{tikzcd}$$

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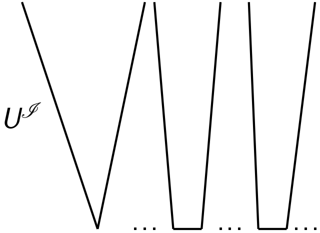

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Examples.

Finite-kernel model. Let U be a model of $\mathrm{ZFCU_R} + \mathscr{A} \sim \omega$ and $\mathscr{I}$ be the finite ideal. In $U^{\mathscr{I}}$, $\mathscr{A}$ is a proper class but every set of $\mathscr{A}$ is finite; hence the DC_ω-scheme fails.

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Countable-kernel model. Let U be a model of $\mathrm{ZFCU_R} + \mathscr{A} \sim \omega_1$ and $\mathscr{I}$ be the countable ideal. In $U^{\mathscr{I}}$, $\mathscr{A}$ is a proper class but every set of $\mathscr{A}$ is countable, so the DC_{ω_1}-scheme fails. But $U^{\mathscr{I}} \models$ Collection because $U^{\mathscr{I}} \models$ Tail.

The hierarchy in $\mathsf{ZFU_R}$

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Without AC, different formulations of Plenitude and Tail come apart.

(Plenitude) Every well-ordered cardinal is realized.

(Plenitude$^+$) Every set x is realized.

(Tail) Every set of urelements has a tail.

(Tail*) For every set of urelements, there is a greatest cardinal κ such that κ-many urelements are disjoint from it.

(Tail$^+$) Every set of urelements has a well-ordered tail.

The hierarchy in $\mathrm{ZFU_R}$

Theorem

Over ZFU_R,

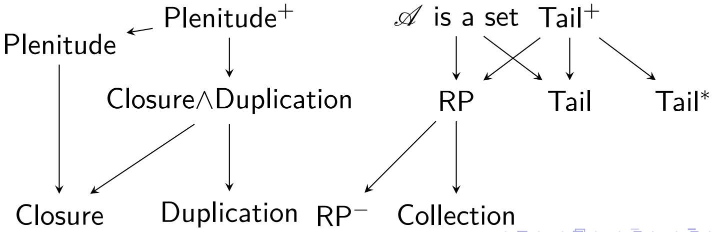
- *Plenitude*$^+$ $\rightarrow$ *Duplication;*
- *Plenitude* $\not\rightarrow$ *(Duplication* $\vee$ *Collection);*
- *(Plenitude* $\wedge$ *Duplication)* $\not\rightarrow$ *Collection;*
- *Tail*$^+$ $\rightarrow$ *RP;*
- *Tail** $\not\rightarrow$ *(Collection* $\vee$ *Tail).*

The hierarchy in ZFU_R

Theorem

Over ZFU_R,

- *Plenitude$^+$ $\rightarrow$ Duplication;*
- *Plenitude $\nrightarrow$ (Duplication $\vee$ Collection);*
- *(Plenitude $\wedge$ Duplication) $\nrightarrow$ Collection;*
- *Tail$^+$ $\rightarrow$ RP;*
- *Tail* $\nrightarrow$ (Collection $\vee$ Tail).*



Open Questions

- $\mathrm{ZFU}_R + \mathit{Plenitude}^+ \vdash \mathit{Collection}$?
- $\mathrm{ZFU}_R + \mathit{Tail} \vdash \mathit{Collection}$?
- $\mathrm{ZFU}_R + \mathit{Collection} \vdash \mathit{RP}^-$?
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- However, models of ZFU_R seem to satisfy the iterative conception of set, i.e., $\forall x\, x \in V(ker(x))$.
- With urelements, the iterative conception of set is weaker than the reflective conception.
- With AC, ZFU_R + Collection seems to be a robust theory (more on this later).

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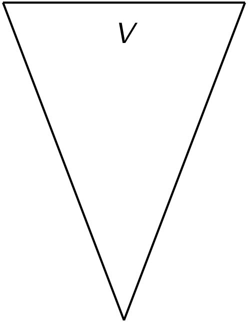
Forcing in ZF



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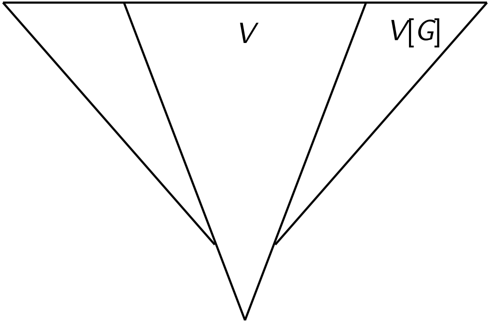

Forcing in ZF

In ZF, given a forcing poset $\mathbb{P}$. $V^{\mathbb{P}}$ is the class of $\mathbb{P}$-names consisting sets of pairs $\langle \dot{x}, p \rangle$, where $\dot{x} \in V^{\mathbb{P}}$ and $p \in \mathbb{P}$.

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Given a countable transitive model M of ZF and $\mathbb{P} \in M$, every M-generic filter G over $\mathbb{P}$ produces a generic extension $M[G] = \{\dot{x}_G : \dot{x} \in M^{\mathbb{P}}\}$, where $\dot{x}_G = \{\dot{y}_G : \exists p(\langle \dot{y}, p \rangle \in \dot{x} \wedge p \in G)\}$.

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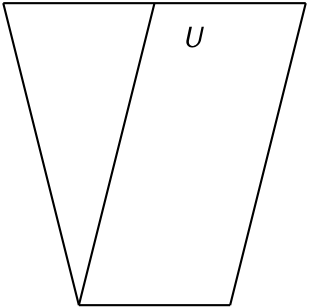
The Forcing Theorem: whatever is true in $M[G]$ is forced by some $p \in G$, and *vice versa*.

Forcing with urelements



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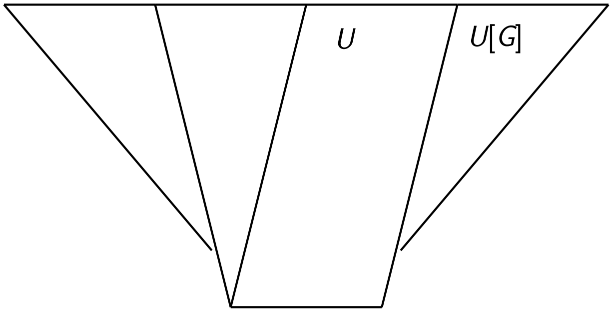

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Definition

Let $\mathbb{P}$ be a forcing poset. $\dot{x} \in U_{\#}^{\mathbb{P}}$ iff either $\dot{x}$ is an urelement, or $\dot{x}$ is a set of ordered-pairs $\langle \dot{y}, p \rangle$, where $\dot{y} \in U_{\#}^{\mathbb{P}}$. For every $p \in \mathbb{P}$ and $\dot{x}, \dot{y} \in U^{\mathbb{P}}$,

- $p \Vdash_{\#} \dot{x} = \dot{y}$ iff ($\dot{x}$ and $\dot{y}$ are the same urelement) $\vee$ $(p \Vdash_{\#} \dot{x} \subseteq \dot{y} \wedge \dot{y} \subseteq \dot{x})$;
- $p \Vdash_{\#} \mathscr{A}(\dot{x})$ iff $\dot{x}$ is an urelement.

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Blass and Ščedrov adopted the following approach, treating each urelement as a different copy of $\emptyset$.

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Let $\mathbb{P}$ be a forcing poset. $\dot{x} \in U^{\mathbb{P}}_{\#}$ iff either $\dot{x}$ is an urelement, or $\dot{x}$ is a set of ordered-pairs $\langle \dot{y}, p \rangle$, where $\dot{y} \in U^{\mathbb{P}}_{\#}$. For every $p \in \mathbb{P}$ and $\dot{x}, \dot{y} \in U^{\mathbb{P}}$,

- $p \Vdash_{\#} \dot{x} = \dot{y}$ iff ($\dot{x}$ and $\dot{y}$ are the same urelement) $\vee$ $(p \Vdash_{\#} \dot{x} \subseteq \dot{y} \wedge \dot{y} \subseteq \dot{x})$;
- $p \Vdash_{\#} \mathscr{A}(\dot{x})$ iff $\dot{x}$ is an urelement.

If M is a countable transitive model of $\mathrm{ZFU_R}$ and G is an M-generic filter over $\mathbb{P}$, then for every $\dot{x} \in M^{\mathbb{P}}_{\#}$

$$\dot{x}_G = \begin{cases} \dot{x} & \text{if } \mathscr{A}(x) \\ \{\dot{y}_G : \exists p \in G \langle \dot{y}, p \rangle \in \dot{x}\} & \text{otherwise} \end{cases}$$

$M[G]_{\#} = \{\dot{x}_G : \dot{x} \in M^{\mathbb{P}}_{\#}\}$.

A problem with $\Vdash_{\#}$

Definition

A forcing relation $\Vdash$ is **full** iff whenever $p \Vdash \exists y\, \varphi(y, \dot{x}_1, ..., \dot{x}_n)$, $p \Vdash \varphi(\dot{y}, \dot{x}_1, ..., \dot{x}_n)$ for some $\mathbb{P}$-name $\dot{y}$.

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$\Vdash_{\#}$ is almost *never* full.

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Proof.

Suppose that $\mathbb{P}$ has a a maximal antichain $\langle p_i : i \in I \rangle$ and $\langle a_i : i \in I \rangle$ are some distinct urelements. Let $\dot{x} = \{\langle a_i, p_i \rangle : i \in I\}$. Then $1_{\mathbb{P}} \Vdash_{\#} \exists y\, (y \in \dot{x})$. But no name can witness this as names of urelements cannot be mixed. $\square$

A new approach

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Definition (Wu, Y.)

Let $\mathbb{P}$ be a forcing poset. $\dot{x} \in U^{\mathbb{P}}$ iff

- $\dot{x}$ is a set of ordered-pairs $\langle y, p \rangle$ where $y \in U^{\mathbb{P}}$ or y is an urelement; and
- whenever $\langle a, p \rangle, \langle y, q \rangle \in \dot{x}$, where a is an urelement and $a \neq y$, then p and q are incompatible.

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- When $\langle a, p \rangle \in \dot{x}$, then p is the "degree" to which a is *identical to* $\dot{x}$.
- The incompatibility condition ensures that when $\dot{x}$ is collapsed to an urelement nothing bad happens.
- As a result, we can *mix* different names.

A new approach

Definition

Let $\mathbb{P}$ be a forcing poset, $p \in \mathbb{P}$, and $x_1, \ldots, x_n \in U^{\mathbb{P}}$.

- $p \Vdash \mathscr{A}(\dot{x}_1)$ iff $\{q \in \mathbb{P} : \exists \langle a, r \rangle \in \dot{x}_1 \ (\mathscr{A}(a) \wedge q \leq r)\}$ is dense below p.
- $p \Vdash \dot{x}_1 \overset{\mathscr{A}}{=} \dot{x}_2$ iff $\{q \in \mathbb{P} : \exists a, r_1, r_2 (\mathscr{A}(a) \wedge \langle a, r_1 \rangle \in \dot{x}_1 \wedge \langle a, r_2 \rangle \in \dot{x}_2 \wedge q \leq r_1, r_2)\} \cup \{q \in \mathbb{P} : \forall \langle a_1, r_1 \rangle \in \dot{x}_1 \ (\mathscr{A}(a_1) \rightarrow q \perp r_1) \wedge \forall \langle a_2, r_2 \rangle \in \dot{x}_2 \ (\mathscr{A}(a_2) \rightarrow q \perp r_2)\}$ is dense below p.
- $p \Vdash \dot{x}_1 = \dot{x}_2$ iff $p \Vdash \dot{x}_1 \subseteq \dot{x}_2 \wedge \dot{x}_2 \subseteq \dot{x}_1 \wedge \dot{x}_1 \overset{\mathscr{A}}{=} \dot{x}_2$.

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The following are equivalent over ZFCU_R.

- *Collection.*
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Note. Another reason to think that $\mathrm{ZFCU_R}$ + Collection is a natural theory.

Generic extensions

Definition

Let M be a ctm of $\mathrm{ZFU_R}$, $\mathbb{P} \in M$ and G be an M-generic filter over $\mathbb{P}$. For every $\dot{x} \in M^{\mathbb{P}}$,

- $\dot{x}_G = a$ if $\mathscr{A}(a)$ and $\langle a, p \rangle \in \dot{x}$ for some $p \in G$;
- $\dot{x}_G = \{\dot{y}_G : \langle \dot{y}, p \rangle \in \dot{x}$ for some $\dot{y} \in M^{\mathbb{P}}$ and $p \in G\}$ otherwise.

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$M \subseteq M[G]$. $G \in M[G]$. $\mathscr{A}^M = \mathscr{A}^{M[G]}$. $M[G] = M[G]_{\#}$.

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The Forcing Theorem

$(p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_n))^M$ iff for every G containing p, $M[G] \models \varphi(\dot{x}_{1G}, \ldots, \dot{x}_{nG})$.

Axiom preservation

Theorem

Forcing over ZFU_R *preserves* ZFU_R, *Collection, AC, Plenitude*$(^+)$, *Tail, Duplication,* $DC_{<\mathrm{Ord}}$, RP^-, *and RP. Forcing over* ZFCU_R *preserves Closure.*

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Theorem

κ^+-closed forcing notions preserve the DC_κ-scheme over ZFCU_R.

Ground model definability

Theorem (Laver, Woodin, independently)

Every transitive model of ZFC is definable in all of its generic extensions with parameters.

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Laver's argument (also attributed to Hamkins) can be easily generalized to show that the GMD Theorem holds if the ground model has only a set of urelements.

Theorem

If M is a transitive model of ZFCU_R where some κ is not realized, then M is definable (with parameters) in all of its generic extensions generated by κ-closed forcings.

Containing Lemma

Every set of urelements in $M[G]$ is contained in some set of urelements in M.

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Lemma

In ZFU_R*, every permutation* π *of a set of urelements can be extended to an automorphism of* U*. If* π *point-wise fixes* $ker(x)$*,* $\pi x = x$*.*

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Lemma

In ZFU_R, *every permutation* π *of a set of urelements can be extended to an automorphism of* U. *If* π *point-wise fixes* $ker(x)$, $\pi x = x$.

Proof.

Point-wise fix every urelement not in the set and let $\pi x = \{\pi y : y \in x\}$ if x is a set. □

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Theorem

*Let M be a ctm of ZFU_R + DC_ω-scheme + $\mathscr{A}$ is a proper class. Then M has a generic extension in which M is **not** definable with parameters.*

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Let M be a ctm of ZFU_R + DC_ω*-scheme* + $\mathscr{A}$ *is a proper class. Then M has a generic extension in which M is* ***not*** *definable with parameters.*

Proof.

Let $\mathbb{P} \in M$ be $\mathrm{Fn}(\omega, 2)$ and G be an M-generic filter over $\mathbb{P}$. Suppose *for reductio* that $M = \{x \in M[G] : M[G] \models \varphi(x, \dot{u}_G)\}$.

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By the DC_ω-scheme, there is an infinite set of urelements $B' \in M$ disjoint from $ker(\dot{u})$ and B' contains a new countable subset B in $M[G] \setminus M$. Moreover, there is another countable $C \in M$ disjoint from $ker(\dot{u}) \cup B'$.

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$M[G]$ then has an automorphism that swaps C and B while point-wise fixing $ker(\dot{u})$.Since $M[G] \models \neg\varphi(B, \dot{u}_G)$ and $ker(\dot{u}_G) \subseteq ker(\dot{u})$, it follows that $M[G] \models \neg\varphi(C, \dot{u}_G)$ and hence $C \notin M$, which is a contradiction. $\square$

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Let M be a transitive model of ZFCU_R + *Plenitude. Then M is **not** definable with parameters in **any** of its non-trivial generic extensions.*

Proof.

Suppose that $M \subsetneq M[G]$ and $M = \{x \in M[G] : M[G] \models \varphi(x, \dot{u}_G)\}$. Fix some $\dot{z}_G \in M[G] \setminus M$ such that $\dot{z}_G \subseteq V_\alpha(A)^M$ for some α and $A \in M$.

By Plenitude and AC in M, there is a bijection f from $V_\alpha(A)^M$ to a $B' \in M$. $B = f[\dot{z}_G] \subseteq B'$ will then be a new set of urelements in $M[G]$.

In M, B' has a duplicate disjoint from $ker(\dot{u})$ which has a new subset D.

By AC and Plenitude in M, there a duplicate $C \in M$ of D that is disjoint from $ker(\dot{u})$. This gives an automorphism as before—contradiction. □

Destroy DC_{ω_1}-Scheme

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Lemma

In ZFCU_R, *if every set of urelements has a tail of size at least* κ, *then the* DC_κ*-scheme holds.* □

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Forcing over ZFCU_R + Collection does not preserve the DC_{ω_1}-scheme.

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Forcing over ZFCU_R + Collection does not preserve the DC_{ω_1}-scheme.

Proof.

Let M be a ctm of $\mathrm{ZFCU_R}$ where every set of urelements has an ω_1-tail. Both Collection and the DC_{ω_1}-scheme hold in M.

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In M, let $\mathbb{P} = \mathrm{Col}(\omega, \omega_1)$ and G be M-generic over $\mathbb{P}$.

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In $M[G]$, every set of urelements is countable by the Containing Lemma. So $\mathscr{A}$ is a proper class but every set of urelements is countable, violating the DC_{ω_1}-scheme. □

Observation

Forcing over $\mathsf{ZFCU_R}$ + Collection preserves the DC_ω-Scheme.

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Collection $\rightarrow$ DC_ω-Scheme over $\mathsf{ZFCU_R}$ and forcing preserves Collection. $\square$

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Open Question

Does forcing over ZFCU_R preserve the DC_ω-Scheme?

Axiom resurrection

Theorem

Let M be a countable transitive model of $\mathrm{ZFCU}_R + DC_\omega$-Scheme. Every forcing extension of M has a forcing extension that satisfies the Reflection Principle.

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Let $M[G]$ be a forcing extension of M. By the following part of the diagram,

$$\begin{tikzcd}
\text{Plenitude} \arrow[ddr] & \mathscr{A} \text{ is a set} \arrow[d] & \\
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we may assume that in $M[G]$, $\mathscr{A}$ is a proper class but there is a least cardinal κ not realized.

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Theorem

Let M be a countable transitive model of $\mathrm{ZFCU}_R + DC_\omega$-Scheme. Every forcing extension of M has a forcing extension that satisfies the Reflection Principle.

Proof.

Force with $\mathrm{Col}(\omega, \kappa)$ in $M[G]$ to $M[G][H]$, where every set of urelements becomes countable in $M[G][H]$.

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By Containing, (*) is preserved by forcing and hence holds in $M[G][H]$.

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This shows that every set of urelements in $M[G][H]$ has an ω-tail.

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This shows that every set of urelements in $M[G][H]$ has an ω-tail.

It then follows from the diagram

$$\begin{array}{ccc} \text{Tail} & & \\ \downarrow & & \\ \text{Collection} & \rightarrow & \text{RP} \end{array}$$

that RP holds in $M[G][H]$. □

A potentialist picture

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Thus, from a potentialist point of view, $\mathrm{ZFCU_R} + \mathrm{DC}_\omega$-Scheme is already "good enough" since Collection and RP are shown to be "necessarily forceble" over this theory.

Thank You!