# FORCING WITH URELEMENTS 

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#### Abstract

ZFCU $_{R}$ is ZFC set theory modified to allow a class of urelements. I first isolate a hierarchy of axioms based on $\mathrm{ZFCU}_{\mathrm{R}}$ and argue that the Collection Principle should be included as an axiom in order to obtain a more robust set theory with urelements. I then turn to forcing over countable transitive models of $\mathrm{ZFCU}_{\mathrm{R}}$. A new definition of $\mathbb{P}$-names is given. The resulting forcing relation is full just in case the Collection Principle holds in the ground model. While forcing preserves $\mathrm{ZFCU}_{\mathrm{R}}$ and many axioms in the hierarchy, it can also destroy the $\mathrm{DC}_{\omega_{1}}$-scheme and recover the Collection Principle. The ground model definability fails when the ground model contains a proper class of urelements.


## 1. Introduction

This paper investigates forcing with urelements. The first section introduces the theory $\mathrm{ZFCU}_{\mathrm{R}}$ together with a group of additional axioms. In Section 2, I prove that over $\mathrm{ZFCU}_{\mathrm{R}}$ this group of axioms form a hierarchy in terms of their implication strength. Since $\mathrm{ZFCU}_{\mathrm{R}}$ cannot prove any axiom in this group, in Section 3, I argue that a more robust ZFC set theory with urelements should be formulated with the Collection Principle. In Section 4, I turn to forcing over countable transitive models of $\mathrm{ZFCU}_{\mathrm{R}}$. Forcing with urelements has been studied by Blass and Ščedrov ( [1]), and Hall ([6] and [7]). However, in all existing studies, it is assumed that the ground model has only a set of urelements. Moreover, the standard definition of $\mathbb{P}$-names adopted in the literature has a major drawback in that the corresponding forcing relation is almost never full. Thus I propose a new definition of $\mathbb{P}$-names that overcomes this problem and prove that the resulting forcing relation is full just in case the Collection Principle holds in the ground model. I then prove that forcing over $\mathrm{ZFCU}_{\mathrm{R}}$ preserves $\mathrm{ZFCU}_{\mathrm{R}}$ (in particular, Replacement) and some of the axioms earlier introduced. Forcing is also shown to be able to destroy the $\mathrm{DC}_{\omega_{1}}$-scheme and recover the Collection Principle. Finally, I show that the ground model definability fails badly when the ground model has a proper class of urelements.

Urelements are non-sets over which sets are formed. The language of urelement set theory, in addition to $\in$, contains a unary predicate $\mathscr{A}$ for urelements. $\operatorname{Set}(x)$ abbreviates $\neg \mathscr{A}(x)$. The standard axioms (and axiom schemes) of ZFC, modified to allow urelements, are as follows.

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(Axiom \(\mathscr{A}) \forall x(\mathscr{A}(x) \rightarrow \neg \exists y(y \in x))\).
(Extensionality) \(\forall x, y(\operatorname{Set}(x) \wedge \operatorname{Set}(y) \wedge \forall z(z \in y \leftrightarrow z \in x) \rightarrow x=y)\)
(Foundation) \(\forall x(\exists y(y \in x) \rightarrow \exists z \in x(z \cap x=\emptyset))\)
(Pairing) \(\forall x, y \exists z \forall v(v \in z \leftrightarrow v=x \vee v=y)\)
(Union) \(\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(z \in w))\).
(Powerset) \(\forall x \exists y \forall z(z \in y \leftrightarrow \operatorname{Set}(z) \wedge z \subseteq x)\)
(Separation) \(\forall x, u \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z, u))\)
(Infinity) \(\exists s(\exists y \in s(\operatorname{Set}(y) \wedge \forall z(z \notin y)) \wedge \forall x \in s(x \cup\{x\} \in s))\)
(Replacement) \(\forall w, u(\forall x \in w \exists!y \varphi(x, y, u) \rightarrow \exists v \forall x \in w \exists y \in v \varphi(x, y, u))\)
(AC) Every set is well-orderable.
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## Definition 1.

$\mathrm{ZU}=$ Axiom $\mathscr{A}+$ Extensionality + Foundation + Pairing + Union + Powerset + Infinity + Separation.
$\mathrm{ZFU}_{\mathrm{R}}=\mathrm{ZU}+$ Replacement.
$\mathrm{ZFCU}_{\mathrm{R}}=\mathrm{ZFU}_{\mathrm{R}}+\mathrm{AC}$.
Note that our formulation of $\mathrm{ZFCU}_{\mathrm{R}}$ allows a proper class of urelements. And for this reason, as we shall see, $\mathrm{ZFCU}_{\mathrm{R}}$ is a very weak theory as it cannot prove many ZFC theorems. However, since ZFCU R still suffices for the basic forcing machinery, it serves as a natural starting point for our investigation.

Let us mention some basic facts about $\mathrm{ZFU}_{\mathrm{R}}$. Every object $x$ has a kernel, $\operatorname{ker}(x)$, which is the set of the urelements in the transitive closure of $\{x\}$. The kernel of a urelement is then its singleton, which is somewhat nonstandard but will be useful for our purpose. A set is pure if its kernel is empty. $V$ denotes the class of all pure sets. Ord is the class of all ordinals, which are transitive pure sets well-ordered by the membership relation. For any set of urelements $A$, by transfinite recursion on $\operatorname{Ord}$ we define the $V_{\alpha}(A)$ hierarchy as follows.

$$
\begin{aligned}
& V_{0}(A)=A \\
& V_{\alpha+1}(A)=P\left(V_{\alpha}(A)\right) \cup V_{\alpha}(A) \\
& V_{\gamma}(A)=\bigcup_{\alpha<\gamma} V_{\alpha}(A), \text { where } \gamma \text { is a limit; } \\
& V(A)=\bigcup_{\alpha \in O_{r d}} V_{\alpha}(A)
\end{aligned}
$$

We use $U$ to denote the class of all objects and $\mathscr{A}$ to denote the class of all urelements. $A \subseteq \mathscr{A}$ thus means " $A$ is a set of urelements". For every $x$ and set $A \subseteq \mathscr{A}, x \in V(A)$ if and only if $k e r(x) \subseteq A$. Every permutation $\pi$ of a set of urelements can be extended to a definable permutation of $\mathscr{A}$ by letting $\pi$ be identity elsewhere, and $\pi$ can be further extended to a permutation of $U$ by letting $\pi x$ be $\{\pi y: y \in x\}$ for every set $x$. Such $\pi$ preserves $\in$ and thus is an automorphism of $U$. For every $x, \pi x=x$ whenever $\pi$ point-wise fixes $\operatorname{ker}(x)$. Finally, it is folklore that $\mathrm{ZFCU}_{\mathrm{R}}$ cannot prove the Collection Principle (see Theorem 18).
(Collection) $\forall w, u(\forall x \in w \exists y \varphi(x, y, u) \rightarrow \exists v \forall x \in w \exists y \in v \varphi(x, y, u))$.
However, $\mathrm{ZFU}_{\mathrm{R}}$ proves the following restricted version of Collection.
(Collection $\left.{ }^{-}\right) \forall w, u(\exists A \subseteq \mathscr{A} \forall x \in w \exists y \in V(A) \varphi(x, y, u) \rightarrow \exists v \forall x \in w \exists y \in v \varphi(x, y, u))$.
Proposition 2. $\mathrm{ZFU}_{\mathrm{R}} \vdash$ Collection $^{-}$.
Proof. For every $x \in w$, let $\alpha_{x}$ be the least $\alpha$ such that there is some $y \in V_{\alpha}(A)$ with $\varphi(x, y, u)$ and let $\alpha=\bigcup_{x \in w} \alpha_{x} . V_{\alpha}(A)$ is the desired set $v$.
1.1. Reflection. In pure set theory, the reflection principle is a scheme asserting that any statement $\varphi$ will become absolute between $V$ and an initial fragment of $V$. In particular, ZF proves the following LévyMontague reflection principle.

For every $\alpha$, there is $\beta>\alpha$ such that $\forall x_{1}, \ldots, x_{n} \in V_{\beta}\left(\varphi\left(x_{1}, \ldots, x_{n}\right) \leftrightarrow \varphi^{V_{\beta}}\left(x_{1}, \ldots, x_{n}\right)\right)$.
In urelement set theory, one cannot expect the Lévy-Montague reflection to hold, e.g., if there is a proper class of urelements, then no $V_{\alpha}(A)$ can reflect such statement for any set of urelements $A$. Thus, in the presence of urelements it should be transitive sets that reflect. Namely,
(RP) For every $x$ there is a transitive set $t$ extending $x$ such that for every $v_{1}, \ldots, v_{n} \in t, \varphi\left(v_{0}, \ldots, v_{n}\right) \leftrightarrow$ $\varphi^{t}\left(v_{1}, \ldots, v_{n}\right)$.
We may also consider a seemingly weaker version of RP, which asserts that any true statement is true in some transitive set containing the parameters.
$\left(\mathrm{RP}^{-}\right)$If $\varphi\left(v_{1}, \ldots, v_{n}\right)$, then there is a transitive set $t$ containing $v_{1}, \ldots v_{n}$ such that $\varphi^{t}\left(v_{0}, \ldots, v_{n}\right)$.
This form of reflection was first introduced by Lévy [13]. And in [15] Lévy and Vaught showed that over Zermelo set theory, $\mathrm{RP}^{-}$does not imply RP.
1.2. Dependent choice scheme. The Dependent Choice scheme (studied in [5] and [4]), as a class version of the Axiom of Dependent Choice (DC), asserts that if $\varphi$ defines a class relation without terminal nodes, then there is an infinite sequence threading this relation.
(DC-scheme) If for every $x$ there is some $y$ such that $\varphi(x, y, u)$, then for every $p$ there is an infinite sequence $s$ such that $s(0)=p$ and $\varphi(s(n), s(n+1), u)$ for every $n<\omega$.
Similarly, we can formulate a class version of $\mathrm{DC}_{\kappa}$ for any infinite cardinal $\kappa$.
( $\mathrm{DC}_{\kappa}$-scheme) If for every $x$ there is some $y$ such that $\varphi(x, y, u)$, then there is some function $f$ on $\kappa$ such that $\varphi(f \upharpoonright \alpha, f(\alpha), u)$ for every $\alpha<\kappa$.
$\mathrm{DC}_{<\text {Ord }}$ holds just in case the $\mathrm{DC}_{\kappa}$-scheme holds for every $\kappa$. It is not hard to verify that the $\mathrm{DC}_{\omega}$-scheme is indeed a reformulation of the DC -scheme. The $\mathrm{DC}_{\kappa}$-scheme is equivalent to the scheme that for every definable class $X$, if for every $s \in X^{<\kappa}$ there is some $y \in X$ with $\varphi(x, y, u)$, then there is some function $f: \kappa \rightarrow X$ such that $\varphi(f\lceil\alpha, f(\alpha), u)$ for every $\alpha<\kappa$. It is proved in [5] that over ZFC without Powerset, Collection and the $\mathrm{DC}_{\omega}$-scheme jointly imply RP. The same argument goes through in $\mathrm{ZFU} \mathrm{R}_{\mathrm{R}}$ as well.
Theorem 3 ([5]). $\mathrm{ZFU}_{\mathrm{R}} \vdash$ Collection $\wedge \mathrm{DC}_{\omega}$-scheme $\rightarrow \mathrm{RP}$.

### 1.3. Urelement axioms and homogeneity.

Definition 4. Let $\kappa$ be a cardinal and $A$ be a set of urelements.
(1) $\kappa$ is realized by $A$ if $A$ is equinumerous with $\kappa$;
(2) $\kappa$ is the tail cardinal of $A$ if $\kappa$ is the greatest cardinal realized by some $B \subseteq \mathscr{A}$ disjoint from $A$.

We shall consider the following axioms based on these definitions.
(Plenitude) Every cardinal $\kappa$ is realized.
(Scatter) For every set $A$ of urelements, there is a $B \subseteq \mathscr{A}$ such that $B$ is equinumerous with $A$ and disjoint from $A$.
(Closure) If $X$ is a set of realized cardinals, the supremum of $X$ is realized.
(Tail) Every set of urelements has a tail cardinal.
Another two important notions, first defined in [8], are duplication and homogeneity.
Definition 5. Let $A$ be a set of urelements.
(1) Duplication holds over $A$ if for every $B \subseteq \mathscr{A}$ disjoint from $A$, there is a $C \subseteq \mathscr{A}$ such that $C$ is equinumerous with $B$ and $C$ is disjoint from $A \cup B$;
(2) Homogeneity holds over $A$ if whenever $B$ and $C$ are two equinumerous sets of urelements that are disjoint from $A$, there is an automorphism $\pi$ such that $\pi B=C$ and $\pi$ point-wise fixes $A$.

Intuitively, when homogeneity holds over $A$, then $A$ has contained all the important information so that the urelemnets outside $A$ are all indistinguishable from the perspective of $A$. The following lemma restates and generalizes several observations made in [8].

Lemma $6\left(\mathrm{ZFU}_{\mathrm{R}}\right)$.
(1) If $A \subseteq A^{\prime} \subseteq \mathscr{A}$ and duplication holds over $A$, then duplication holds over $A^{\prime}$.
(2) If duplication holds over $A \subseteq \mathscr{A}$, then homogeneity holds over $A$.
(3) If Tail holds, then duplication holds over some set of urelements.
(4) Assume that every set of urelements is well-orderable. For every $A \subseteq \mathscr{A}$, there is an $A^{\prime} \subseteq \mathscr{A}$ such that $A \subseteq A^{\prime}$ and duplication (hence homogeneity) holds over $A^{\prime}$.
Proof. (1) If $B$ is disjoint from $A^{\prime}$, then there is another $C$ disjoint from $A$ that is equinumerous with $\left(A^{\prime} \backslash\right.$ $A) \cup B$. So there is a $C^{\prime} \subseteq C$ disjoint from $A^{\prime}$ that is equinumerous with $B$.
(2) Let $B$ and $C$ be two sets of urelements that are disjoint from $A$. If they are disjoint, then by swapping them we can get a permutation $\pi$ with $\pi B=C$ that point-wise fixes $A$. If not, then by duplication over $A$, there is another set of urelements disjoint from $A \cup B \cup C$ that is equinumerous with $B \cup C$. This gives us a $D$ that is equinumerous with both $B$ and $C$ and disjoint from $A \cup B \cup C$. Thus, there are permutations $\pi_{1}$ and $\pi_{2}$ such that $\pi_{1} B=D$ and $\pi_{2} D=C$, and both of them point-wise fix $A$. The composition of $\pi_{1}$ and $\pi_{2}$ is the desired permutation.
(3) Let $\kappa$ be the least tail cardinal held by some $D$. We claim that duplication holds over $D$. If $B$ is disjoint from $D$, since the tail cardinal of $D \cup B$ is at least $\kappa$, there is another $C$ disjoint from $D \cup B$ that has size at least $\kappa$. In particular, $C$ contains a subset that is equinumerous with $B$.
(4) Assume that every set of urelements is well-orderable. By (1) and (2), it suffices to show that duplication holds over some set of urelements. Suppose otherwise. Then $\mathscr{A}$ is a proper class; by (3), it follows that some $A \subseteq \mathscr{A}$ has no tail cardinal. Given any infinite $B$ disjoint from $A$, since $B$ is equinumerous with some cardinal $\kappa$, there must be some $C$ of size $\kappa^{+}$that is disjoint from $A$; so there is some $C^{\prime} \subseteq C$ of size $\kappa$ that is disjoint from $A \cup B$. This shows that duplication holds over $A$ after all, which is a contradiction.

The converse of (2) does not hold: if $\mathscr{A}$ is a set and $\mathscr{A} \backslash A$ has only one urelement, then duplication does not hold over $A$ while homogeneity trivially holds over $A$. The assumption that every set of urelements is well-orderable is necessary for (4): as shown in the author's dissertation [17], it is consistent with $\mathrm{ZFU}_{\mathrm{R}}+$ $\mathrm{RP}+\mathrm{DC}_{\omega}$-scheme that homogeneity holds over no set of urelements.

## 2. A HIERARCHY OF AXIOMS OVER $\mathrm{ZFCU}_{R}$

Theorem 7. Over $\mathrm{ZFCU}_{\mathrm{R}}$, the following implication diagram holds. The diagram is also best possible: if the diagram does not indicate that $\varphi$ implies $\psi$, then $\mathrm{ZFCU}_{\mathrm{R}}+\varphi \nvdash \psi$ if $\mathrm{ZFCU}_{\mathrm{R}}$ is consistent.


The direction from Collection to the $\mathrm{DC}_{\omega}$-scheme was first proved by Schlutzenberg in an answer to a question on Mathoverflow [9] and the notion of tail cardinal was also implicit in his proof ([8] also contains of a proof of this fact). Our proof of this fact takes a different route and appeals to the lemma that Tail implies Collection, which is crucial for the later discussion of forcing.

Let us first show that Plenitude implies $\mathrm{DC}_{<\text {Ord }}$. Given a formula $\varphi(x, y, u)$ with a parameter $u$, for any ordinals $\alpha, \alpha^{\prime}, \kappa, \kappa^{\prime}$ and a set of urelements $E$, we say that $\left\langle\kappa^{\prime} \alpha^{\prime}\right\rangle$ is a $(\varphi, E)$-extension of $\langle\kappa, \alpha\rangle$ if (i) $\alpha \leq \alpha^{\prime}$, and (ii) whenever $A \subseteq \mathscr{A}$ extends $E$ by $\kappa$-many urelements, there is some $B \subseteq \mathscr{A}$ disjoint from $A$ with $|B|=\kappa^{\prime}$ such that for every $x \in V_{\alpha}(A)$, there is some $y \in V_{\alpha^{\prime}}(A \cup B)$ such that $\varphi(x, y, u)$.

Lemma 8. $\left(\mathrm{ZFCU}_{\mathrm{R}}\right)$ Suppose that Plenitude holds and $\varphi(x, y, u)$ defines a relation without terminal nodes. Then every $\langle\kappa, \alpha\rangle$ has a $(\varphi, \operatorname{ker}(u))$-extension.

Proof. Note that in this case homogeneity holds over every set of urelements. Fix $\langle\kappa, \alpha\rangle$ and some $A \subseteq \mathscr{A}$ extending $\operatorname{ker}(u)$ with $\kappa$-many urelements. For each $x \in V_{\alpha}(A)$, define $\theta_{x}$ to be the least cardinal such that there is some $y$ with $\varphi(x, y, u)$ and $|\operatorname{ker}(y)|=\theta_{x}$, and let $\kappa^{\prime}=\operatorname{Sup}\left\{\theta_{x}: x \in V_{\alpha}(A)\right\}$. Fix some infinite $B$ of size $\kappa^{\prime}$ that is disjoint from A, which exists by Plenitude. Then for every $x \in V_{\alpha}(A)$, fix some $y^{\prime}$ such that $\varphi\left(x, y^{\prime}, u\right)$ and $\left|\operatorname{ker}\left(y^{\prime}\right)\right|=\theta_{x} . \operatorname{ker}\left(y^{\prime}\right) \backslash A$ is equinumerous to a subset of $B$, so by homogeneity over $A$, there is an automorphism $\pi$ that moves $\operatorname{ker}\left(y^{\prime}\right)$ into $B$ and point-wise fixes $A$. It follows that $\varphi\left(x, \pi y^{\prime}, u\right)$ and $\pi y^{\prime} \in V(A \cup B)$. Thus, each $x \in V_{\alpha}(A)$ has some $y \in V(A \cup B)$ with $\varphi(x, y, u)$, so there is some large enough $\alpha^{\prime}$ such that every $x \in V_{\alpha}(A)$ has some $y \in V_{\alpha^{\prime}}(A \cup B)$ with $\varphi(x, y, u)$. Furthermore, for every $A^{\prime}$ extending $\operatorname{ker}(u)$ by $\kappa$-many urelements, by homogeneity over $\operatorname{ker}(u)$, there is an automorphism $\pi$ with $\pi A=A^{\prime}$ that point-wise fixes $\operatorname{ker}(u)$; so $\pi B$ will be such that every $x \in V_{\alpha}\left(A^{\prime}\right)$ has some $y \in V_{\alpha^{\prime}}\left(A^{\prime} \cup \pi B\right)$ with $\varphi(x, y, u)$. Therefore, $\left\langle\kappa^{\prime}, \alpha^{\prime}\right\rangle$ is indeed a $(\varphi, \operatorname{ker}(u))$-extension of $\langle\kappa, \alpha\rangle$.

Theorem 9. $\mathrm{ZFCU}_{\mathrm{R}} \vdash$ Plenitude $\rightarrow \mathrm{DC}_{<\text {Ord }}$.
Proof. Suppose that Plenitude holds and $\varphi(x, y, u)$ defines a relation without terminal nodes with some parameter $u$. Consider any infinite cardinal $\kappa$. To prove the $\mathrm{DC}_{\kappa}$-scheme, we first find a set $\bar{x}$ that is closed under $<\kappa$-sequences and $\varphi$; we can then apply $\mathrm{DC}_{\kappa}$ to get a desired function on $\kappa$. Let $\delta$ be a cardinal with $\operatorname{cf}(\boldsymbol{\delta})=\kappa$. We first define a $\delta$-sequence of pairs of ordinals $\left\langle\left\langle\lambda_{\alpha}, \gamma_{\alpha}\right\rangle: \alpha<\boldsymbol{\delta}\right\rangle$ by recursion as follows. Let $A_{0}$ be a set of urelements that extends $\operatorname{ker}(u)$ by $\lambda_{0}$-many urelements and $\gamma_{0}$ be an ordinal with $\operatorname{cf}\left(\gamma_{0}\right) \geq \kappa$. For each ordinal $\alpha<\delta$, we let $\left\langle\lambda_{\alpha+1}, \gamma_{\alpha+1}\right\rangle$ be the lexicographical-least $(\varphi, \operatorname{ker}(u))$-extension of $\left\langle\lambda_{\alpha}, \gamma_{\alpha}\right\rangle$ with $\operatorname{cf}\left(\gamma_{\alpha}\right) \geq \kappa$, which exists by the previous lemma. And we take the union at the limit stage.

By Plenitude, we can fix a $\delta$-sequence of sets of urelements $\left\langle A_{\alpha}: \alpha<\delta\right\rangle$, where $A_{\alpha}$ extends $\bigcup_{\beta<\alpha} A_{\beta} \cup$ $\operatorname{ker}(u)$ by $\lambda_{\alpha}$-many urelements. Let $\bar{x}=\bigcup_{\alpha<\delta} V_{\gamma_{\alpha}}\left(A_{\alpha}\right)$. For any $x \in V_{\gamma_{\alpha}}\left(A_{\alpha}\right)$, There is some $B$ disjoint from $A_{\alpha}$ witnessing the fact that $\left\langle\lambda_{\alpha+1}, \gamma_{\alpha+1}\right\rangle$ is a $\left(\varphi, \operatorname{ker}(u)\right.$ )-extension of $\left\langle\lambda_{\alpha}, \gamma_{\alpha}\right\rangle$. And by homogeneity over $A$, it follows that $A_{\alpha+1} \backslash A_{\alpha}$ works as such witness as well; so there is some $y \in V_{\gamma_{\alpha+1}}\left(A_{\alpha+1}\right)$ with $\varphi(x, y, u)$, and such $y$ lives in $\bar{x}$. $\bar{x}$ is also closed under $<\kappa$-sequences since $\operatorname{cf}(\boldsymbol{\delta})=\kappa$ and each $V_{\gamma_{\alpha}}\left(A_{\alpha}\right)$ is closed under $<\kappa$-sequences. Thus, if $s \in \bar{x}^{<\kappa}$, there is some $y \in \bar{x}$ such that $\varphi(s, y, u)$. $\mathrm{By}_{\mathrm{DC}}^{\kappa}$, there exists a function $f$ on $\kappa$ such that $\varphi(f \upharpoonright \alpha, f(\alpha), u)$ for all $\alpha<\kappa$. Hence, the $\mathrm{DC}_{\kappa}$-scheme holds.

## Lemma 10. $\mathrm{ZFCU}_{\mathrm{R}} \vdash$ Closure $\wedge$ Scatter $\rightarrow$ Collection

Proof. Fix some set $w$ such that $\forall x \in w \exists y \varphi(x, y, u)$. For every $x \in w$, let $\theta_{x}$ be the least $\theta$ realized by the kernel of some $y$ such that $\varphi(x, y, u)$, and set $\theta$ as the supremum of all such $\theta_{x}$. Let $A \subseteq \mathscr{A}$ be such that $\operatorname{ker}(w) \cup \operatorname{ker}(u) \subseteq A$ and duplication holds over $A$. By Closure and Scatter, there is a $B \subseteq \mathscr{A}$ of size $\theta$ that is disjoint from $A$. Then for every $x \in w$, fix a $y^{\prime}$ such that $\varphi\left(x, y^{\prime}, u\right)$ with the smallest kernel. By homogeneity over $A$, there is an autormophism that moves $\operatorname{ker}\left(y^{\prime}\right)$ into $A \cup B$ without moving any urelements in $A$. Therefore, every $x \in w$ has a $y \in V(A \cup B)$ such that $\varphi(x, y, u)$. Then Collection holds by applying Proposition 2

## Lemma 11. $\mathrm{ZFCU}_{\mathrm{R}} \vdash$ Tail $\rightarrow$ Collection

Proof. Assume that every set of urelements has a tail cardinal. Suppose that $\forall x \in w \exists y \varphi(x, y, u)$ for some $w$ and $u$. Let $A \subseteq \mathscr{A}$ be such that $\operatorname{ker}(w) \cup \operatorname{ker}(u) \subseteq A$ and duplication holds over $A$, and let $\kappa$ be the tail cardinal of $A$. Fix some $B \subseteq \mathscr{A}$ disjoint from $A$ that has size $\kappa$. For every $x \in w$ and $y$ such that $\varphi(x, y, u), B$ must contain a subset that is equinumerous with $\operatorname{ker}(y) \backslash A$; by homogeneity over $A$, there is an automorphism that moves $\operatorname{ker}(y)$ into $A \cup B$ without moving any urelements in $A$. Therefore, every $x \in w$ has some $y \in V(A \cup B)$ such that $\varphi(x, y, u)$ and hence Collection holds by Proposition 2 .

Lemma 12. $\mathrm{ZFCU}_{\mathrm{R}} \vdash$ Tail $\rightarrow \mathrm{DC}_{\omega}$-scheme
Proof. Assume that every set of urelements has a tail cardinal. Suppose that $\varphi(x, y, u)$ defines a relation without terminal nodes with a parameter $u$ and fix some $p$. We wish to construct a set containing $p$ that is closed under the relation $\varphi$, and then we can apply DC to get the desired $\omega$-sequence. Let $A$ be a set of urelements extending $\operatorname{ker}(p) \cup \operatorname{ker}(u)$ over which duplication holds and $\kappa$ be the tail cardinal of $A$.

Claim 12.1. Every $\langle\kappa, \alpha\rangle$ has a $(\varphi, A)$-extension $\left\langle\kappa, \alpha^{\prime}\right\rangle$
Proof of the Claim. If $B$ extends $A$ by $\kappa$-many urelements, then by duplication over $A$, there will be another $C$ of size $\kappa$ that is disjoint from $A \cup B$. Then for every $x \in V_{\alpha}(B)$ and $y$ with $\varphi(x, y, u), \operatorname{ker}(y) \backslash B$ must be equinumerous with some subset of $C$ since $\kappa$ is the tail cardinal of $A$. By homogeneity over $B$, there is an automorphism that moves $\operatorname{ker}(y)$ into $C$ without moving any urelements in $B$. So by taking a sufficiently large $\alpha^{\prime}$, it follows that for every $x \in V_{\alpha}(B)$ there is some $y \in V_{\alpha^{\prime}}(B \cup C)$ such that $\varphi(x, y, u)$. Furthermore, for every $B^{\prime}$ that extends $A$ by $\kappa$-many urelements, by homogeneity over $A$, there is a corresponding $C^{\prime}$ such that every $x \in V_{\alpha}\left(B^{\prime}\right)$ has some $y \in V_{\alpha^{\prime}}\left(B^{\prime} \cup C^{\prime}\right)$ with $\varphi(x, y, u)$.

Now let $\alpha_{0}$ be some large enough $\alpha$ such that $u \in V_{\alpha_{0}}(A)$. We construct a sequence $\left\langle\alpha_{n}: n<\omega\right\rangle$ by letting $\alpha_{n+1}$ be the least ordinal such that $\left\langle\kappa, \alpha_{n+1}\right\rangle$ is a $(\varphi, A)$-extension of $\left\langle\kappa, \alpha_{n}\right\rangle$ with respect to $A$. Then fix a sequence of sets of urelements $\left\langle A_{n}: n<\omega\right\rangle$ such that each $A_{n}$ extends $\bigcup_{m<n} A_{m} \cup A$ by $\kappa$-many urelements. Such sequence exists because $A$ has tail cardinal $\kappa$ and every $B$ of size $\kappa$ can be partitioned into infinitely many pieces of size $\kappa$. Then set $x=\bigcup_{n<\omega} V_{\alpha_{n}}\left(A_{n}\right)$. For every $x \in V_{\alpha_{n}}\left(A_{n}\right)$, by homogeneity over $A_{n}$ it follows that there is some $y \in V_{\alpha_{n+1}}\left(A_{n+1}\right)$ such that $\varphi(x, y, u)$. Hence, $x$ is a set containing $u$ that is closed under $\varphi(x, y, u)$. This completes the proof.

Lemma $13\left(\mathrm{ZFCU}_{\mathrm{R}}\right)$. Let $\kappa$ be a cardinal and suppose that every set of urelements has a tail cardinal which is at least $\kappa$. Then the $\mathrm{DC}_{\kappa}$-scheme holds.

Proof. First assume that $\kappa$ is regular. Suppose that $\varphi(x, y, u)$ defines a relation without terminal nodes with a parameter $u$. We wish to construct a set $x$ such that for every $s \in x^{<\kappa}$, there is some $y \in x$ with $\varphi(s, y, u)$ and then apply $\mathrm{DC}_{\kappa}$ to obtain the desired sequence.

Let $A$ be a set of urelements extending $\operatorname{ker}(u)$ over which duplication holds. By a similar argument as before, we see that every $\langle\kappa, \alpha\rangle$ has a $(\varphi, A)$-extension $\left\langle\kappa, \alpha^{\prime}\right\rangle$, where $\alpha^{\prime}$ can be arbitrarily large. And we can define a sequence of ordinals $\left\langle\gamma_{\alpha}: \alpha<\kappa\right\rangle$ by recursion, where $\gamma_{\alpha}$ is the least ordinal such that $\left\langle\kappa, \gamma_{\alpha}\right\rangle$ is a a $(\varphi, A)$-extension of $\left\langle\kappa, \bigcup_{\beta<\alpha} \gamma_{\beta}\right\rangle$ and $\operatorname{cf}\left(\gamma_{\alpha}\right)=\kappa$. Then fix a sequence of sets of urelements $\left\langle A_{\alpha}: \alpha<\kappa\right\rangle$, where $A_{\alpha}$ extends $\bigcup_{\beta<\alpha} A_{\beta} \cup A$ by $\kappa$-many urelements. Let $x=\bigcup_{\alpha<\kappa} V_{\gamma_{\alpha}}\left(A_{\alpha}\right)$. $x$ is then closed under $\varphi(x, y, u)$. And since $x$ is the union of an increasing $\kappa$-sequence of sets and each $\gamma_{\alpha}$ has cofinality $\kappa$, it follows that $x^{<\kappa} \subseteq x$. Therefore, we can apply $\mathrm{DC}_{\kappa}$ to $x$ to get the desired $\kappa$ sequence.

Suppose $\kappa$ is singular. Then for every regular $\lambda<\kappa$, the argument in the previous paragraph shows that the $\mathrm{DC}_{\lambda}$-scheme holds, and hence the $\mathrm{DC}_{\lambda}$-scheme holds for every $\lambda<\kappa$. But this implies that the $\mathrm{DC}_{\kappa}$-scheme holds by a standard argument as in [10, Theorem 8.1].

To show that the implication diagram holds, it remains to prove the following non-trivial implications.
Lemma 14. Over $\mathrm{ZFCU}_{\mathrm{R}}$, the following implications hold.
(1) $\mathscr{A}$ is a set $\rightarrow \mathrm{DC}_{<O r d}$.
(2) $\mathrm{DC}_{<\text {Ord }} \rightarrow$ Collection
(3) $\mathrm{RP}^{-} \rightarrow$ Collection.
(4) Collection $\rightarrow$ Closure
(5) Collection $\rightarrow \mathrm{DC}_{\omega}$-scheme.
(6) Collection $\rightarrow$ RP.

Proof. (1) This is proved by a standard argument, which I include for completeness. Assume $\mathscr{A}$ is a set and $\forall x \exists y \varphi(x, y, u)$. Fix some cardinal $\kappa$ and we show that the $\mathrm{DC}_{\kappa}$-scheme holds. Let $\delta$ be such that $\operatorname{cf}(\boldsymbol{\delta})=\kappa$. We define a $\delta$-sequence of ordinals $\left\langle\gamma_{\alpha}: \alpha<\delta\right\rangle$ as follows. Let $\gamma_{0}$ be such that $V_{\gamma_{0}}(\mathscr{A})$ is closed under $<\kappa$-sequences. For any $\alpha<\delta$, let $\gamma_{\alpha}$ be the least $\gamma$ such that (i) $V_{\gamma}(\mathscr{A})$ is closed under $<\kappa$-sequences and (ii) $\forall x \in \bigcup_{\beta<\alpha} V_{\gamma_{\beta}}(\mathscr{A}) \exists y \in V_{\gamma}(\mathscr{A}) \varphi(x, y, u)$. Set $\bar{x}=\bigcup_{\alpha<\delta} V_{\gamma_{\alpha}}(\mathscr{A})$. For every $s \in \bar{x}^{<\kappa}$, there is some $y \in \bar{x}$ such that $\varphi(s, y, u)$. By $\mathrm{DC}_{\kappa}$, it follows that there is an $f: \kappa \rightarrow \bar{x}$ such that $\varphi(f \upharpoonright \alpha, f(\alpha), u)$ for all $\alpha<\kappa$. Therefore, the $\mathrm{DC}_{\kappa}$-scheme holds.
(2) This is because under $\mathrm{DC}_{<\text {Ord }}$, either $\mathscr{A}$ is a set or Plenitude holds, but Collection holds either way by Lemma .
(3) Suppose that $\mathrm{RP}^{-}$holds. We may assume that $\mathscr{A}$ is not a set and Plenitude fails by Lemma. It then suffices to show that Tail holds by Lemma 11 . Fix some $A \subseteq \mathscr{A}$. Let $x=\{\lambda: \exists B \subseteq \mathscr{A}(|B|=\lambda \wedge B \cap A=\emptyset)\}$. $x$ is a non-empty set by assumption. Let $\kappa$ be the supremum of $x$. We claim that $\kappa$ is the tail cardinal of $A$. Let $t$ be a transitive set that reflects the statement that $\forall \lambda<\kappa \exists B(|B|=\lambda \wedge B \cap A=\emptyset)$. We may assume that $t$ extends $\{\kappa, A\}$ and is closed under pairs. Then for every $\lambda<\kappa$, there is some $B \in t$ disjoint from $A$ such that $|B|=\lambda$. Let $C=\bigcup\{B \in t: B \cap A=\emptyset\}$. Since $|C|=\kappa$, it follows that $\kappa$ is the tail cardinal of $A$.

Now assume Collection.
(4) To show Closure holds, let $x$ be a set of realized cardinals. Then by Collection, there is a set $y$ such that for every $\kappa \in x$, there is some $A \in y$ such that $|A|=\kappa$. Let $B=\bigcup\{A: A \in y\}$. Then the cardinality of $B$ is at least the supremum of $x$ and hence Closure holds.
(5) First we observe that Collection $+\neg$ Plenitude implies Tail. Given a set $A$ of urelements, let $\bar{x}=\{\lambda$ : $\exists B(B \cap A=\emptyset \wedge|B|=\lambda)\}$. By Collection, there is some $\bar{y}$ such that for every $\lambda \in \bar{x}$, there is some $B \in \bar{y}$ such that $|B|=\lambda$ and $B \cap A=\emptyset$. Let $C=\bigcup\{B \in \bar{y}: B \cap A=\emptyset\}$. The cardinality of $C$ is the tail cardinal of $A$. Now we may assume that Plenitude fails by Theorem 9 so the $\mathrm{DC}_{\omega}$-scheme holds by Lemma 12 .
(6) RP holds by (5) and Theorem 3 .

Many of the proofs above seem to use AC (or the assumption that every set of urelements is well-orderable) in an essential way. In the choiceless context, many questions remain open.

## Open Questions

(1) Is RP provable from $\mathrm{ZFU}_{\mathrm{R}}+$ Collection?
(2) Is RP provable $\mathrm{ZFU}_{R}+\mathrm{RP}^{-}$?
(3) Is the $\mathrm{DC}_{\omega}$-scheme provable from $\mathrm{ZFU}_{\mathrm{R}}+$ Collection +DC ?
(4) Is the $\mathrm{DC}_{\omega}$-scheme provable from $\mathrm{ZFU}_{\mathrm{R}}+$ Collection $+\mathrm{DC}+$ Plenitude?

I now proceed to show that the implication diagram in Theorem 7 is best possible, which uses an easy method of building inner models of $\mathrm{ZFCU}_{\mathrm{R}}$. This method was implicitly used in [14] and [2].

Definition 15. For any set $x, I \subseteq P(x)$ is a normal ideal on $x$ if (i) $x \notin I$; (ii) $I$ is closed under finite union and subset; and (iii) for every $y \in x,\{y\} \in I$. If $\mathscr{A}$ is an infinite set and $I$ is a normal ideal on $\mathscr{A}, U^{I}=\{x \in$ $U: \operatorname{ker}(x) \in I\}$.

Lemma 16. Assume that $\mathscr{A}$ is an infinite set and $I$ is a normal ideal on $\mathscr{A}$. Then for every $a, A$ such that $a \in A \in I$, there is a permutation $\pi$ of $\mathscr{A}$ such that (i) $\pi^{+} I=I$, (ii) $\pi a \neq a$ and (iii) $\forall b \in A \backslash\{a\}(\pi b=b)$.

Proof. Fix some $a^{*} \in \mathscr{A} \backslash A$. Let $\pi$ be a permutation that swaps $a$ and $a^{*}$ but fixes everything else in $\mathscr{A}$. To see that $\pi^{+} I=I$, let $B \in I$. Without lost of generality, we may assume $a \in B$ and $a^{*} \notin B$. Then $\pi^{+} B=(B \backslash\{a\}) \cup\left\{a^{*}\right\}$, which is in $I$ since $I$ is a normal ideal. Also, $B=\pi^{+}\left((B \backslash\{a\}) \cup\left\{a^{*}\right\}\right)$. Therefore, $\pi^{+} I=I$.

Theorem 17. Let $U$ be a model of $\mathrm{ZFCU}_{\mathrm{R}}$ such that $U \models$ " $\mathscr{A}$ has size $\kappa$ " + " $I$ is a normal ideal on $\mathscr{A}$ ", for some infinite cardinal $\kappa$, then $U^{I} \models \mathrm{ZFCU}_{\mathrm{R}}+$ " $\mathscr{A}$ is a proper class".

Proof. It is clear that $U^{I}$ is transitive and contains all the urelements and pure sets. Thus, $U^{I}$ satisfies Foundation, Extensionality, Infinity, and $\mathscr{A}$ is a proper class in $U^{I} . U^{I}$ is also closed under powerset, pairing and union, as these operations do not add urelements or only add very few urelements in the sense of $I$. AC holds in $U^{I}$ because if $x$ is a set in $U^{I}$, then any bijection in $U$ from $x$ to an ordinal has the same kernel as $x$ and hence is also in $U^{I}$. It remains to show that Replacement holds in $U^{I}$.

Suppose that $U^{I} \models \forall x \in w \exists!y \varphi(x, y, u)$ for some $w, u \in U^{I}$. Let $\bar{y}=\left\{y \in U^{I}: \exists x \in w \varphi^{U^{I}}(x, y, u)\right\}$, which is a set in $U$. It suffices to show that $\operatorname{ker}(\bar{y}) \subseteq \operatorname{ker}(w) \cup \operatorname{ker}(u)$. Suppose not. Then there are some $y$ and $a$ such that $y \in \bar{y}, a \in \operatorname{ker}(\{y\})$ and $a \notin \operatorname{ker}(w) \cup \operatorname{ker}(u)$. Let $A=\operatorname{ker}(w) \cup \operatorname{ker}(u) \cup \operatorname{ker}(\{y\})$, which is in $I$. By Lemma 16 , there is an automorphism $\pi$ such that (i) $\pi I=I$, (ii) $\pi a \neq a$ and (iii) $\pi$ pointwise fixes $A$. Since $y \in \bar{y}$, there is some $x \in w$ with $\varphi^{U^{I}}(x, y, u)$. It follows that $\varphi^{U^{I}}(x, \pi y, u)$, but $\pi y \neq y$ because $\pi a$ is in $\operatorname{ker}(\{\pi y\})$ but not in $\operatorname{ker}(\{y\})$, which contradicts the uniqueness of $y$. Therefore, $\operatorname{ker}(\bar{y}) \subseteq$ $\operatorname{ker}(w) \cup \operatorname{ker}(u)$.

Theorem 18. Assume the consistency of $\mathrm{ZFCU}_{\mathrm{R}}$.
(1) $\mathrm{ZFCU}_{\mathrm{R}}+$ Collection + Closure $\wedge$ Scatter $\nvdash$ Plenitude $\vee$ the $\mathrm{DC}_{\omega_{1}}$-scheme;
(2) $\mathrm{ZFCU}_{\mathrm{R}}+$ Collection $\nvdash$ Scatter;
(3) $\mathrm{ZFCU}_{\mathrm{R}}+$ Scatter $\nvdash$ Closure $\vee$ the $\mathrm{DC}_{\omega}$-scheme;
(4) $\mathrm{ZFCU}_{\mathrm{R}}+$ Closure $\nvdash$ the $\mathrm{DC}_{\omega}$-scheme;
(5) For any infinite cardinal $\kappa, \mathrm{ZFCU}_{\mathrm{R}}+$ the $\mathrm{DC}_{\kappa}$-scheme $\nvdash$ Closure.
(6) For any infinite cardinals $\kappa<\lambda, \mathrm{ZFCU}_{\mathrm{R}}+$ Collection + the $\mathrm{DC}_{\kappa}$-scheme $\nvdash$ the $\mathrm{DC}_{\lambda}$-scheme.

Hence, the implication diagram in Theorem 7 is best possible.
Proof. It is folklore that $\mathrm{ZFCU}_{\mathrm{R}}$ is equiconsistent with ZFC , which is in turn equiconsistent with $\mathrm{ZFCU}_{\mathrm{R}}+$ $"|\mathscr{A}|=\kappa$ " for any cardinal $\kappa$ (see [8] for a proof of this).
(1) Assume that in $U,|\mathscr{A}|=\omega_{1}$. Let $I_{1}$ be the ideal of all countable subsets of $\mathscr{A}$. In $U^{I_{1}}, \omega$ is the greatest realized cardinal, so Closure holds and Plenitude fails. And it is clear that Scatter and Tail hold. So Collection holds in $U^{I_{1}}$ by Lemma 11 . The $\mathrm{DC}_{\omega_{1}}$-scheme fails in $U^{I_{1}}$ because every kernel can be properly extended but there cannot be a function $f$ on $\omega_{1}$ such that $\operatorname{ker}(f \upharpoonright \alpha) \subsetneq \operatorname{ker}(f(\alpha))$ for all $\alpha<\omega_{1}$, as the kernel of such $f$ would be uncountable.
(2) Assume that in $U,|\mathscr{A}|=\omega_{2}$. Fix an $A \subseteq \mathscr{A}$ of size $\omega_{1}$. Let $I_{2}=\{B \subseteq \mathscr{A}: B \backslash A$ is countable $\}$. For every $B \in U^{I_{2}}$, let $\lambda=\operatorname{Max}\{|A \backslash B|, \omega\}$. $\lambda$ is the tail cardinal of $B$ because if $D \in U^{I_{2}}$ is disjoint from $B$ but has size $>\lambda$, then $D$ must extend $A$ by $\omega_{1}$-many urelements, which is impossible. Therefore, Tail and hence Collection holds in $U^{I_{2}}$. The failure of Scatter in $U^{I_{2}}$ is witnessed by $A$.
(3) Assume that in $U,|\mathscr{A}|=\omega$. Let $I_{3}$ be the ideal of finite subsets on $\mathscr{A}$. It is cleat that in $U^{I_{3}}$ Scatter holds and Closure fails. The $\mathrm{DC}_{\omega}$-scheme also fails in $U^{I_{3}}$ because set of urelements can be properly extended but there is no infinite increasing sequence of sets of urelements.
(4) Assume that in $U,|\mathscr{A}|=\omega_{1}$ and fix a countably infinite $A \subseteq \mathscr{A}$. Let $I_{4}=\{B \subseteq \mathscr{A}: B \backslash A$ is finite $\}$. Closure holds in $U^{I_{4}}$ because $\omega$ is the greatest realized cardinal. The $\mathrm{DC}_{\omega}$-scheme fails in $U^{I_{4}}$ since every set of urelements can be properly extended by another set of urelements disjoint from $A$. but there cannot be a corresponding infinite sequence.
(5) Let $\kappa$ be an infinite cardinal. Assume that in $U,|\mathscr{A}|=\omega_{\kappa^{+}}$. Let $I_{5}=\left\{B \subseteq \mathscr{A}:|B|<\omega_{\kappa^{+}}\right\}$. Closure fails in $U^{I_{5}}$ because $\omega_{K^{+}}$is not realized while every cardinal below it is realized. To show that the $\mathrm{DC}_{K^{-}}$ scheme holds, suppose that for every $x \in U^{I_{5}}$, there is some $y \in U^{I_{5}}$ such that $\varphi^{U^{I_{5}}}(x, y, u)$. By Lemma 14 . in $U$ there is a function $f: \kappa \rightarrow U^{I_{5}}$ such that $U^{I_{5}} \models \varphi(f \upharpoonright \alpha, f(\alpha), u)$ for every $\alpha<\kappa$. Since $k e r(f)=$
$\bigcup_{\alpha<\kappa} \operatorname{ker}(f(\alpha))$, and the kernel of each $f(\alpha)$ has size less than $\omega_{\kappa^{+}}, \operatorname{ker}(f)$ has size less than $\omega_{\kappa^{+}}$. Hence, $f$ is in $U^{I_{5}}$.
(6) It suffices to show that for any $\kappa, \mathrm{ZFCU}_{\mathrm{R}}+$ Collection + the $\mathrm{DC}_{\kappa^{-}}$-scheme does not prove the $\mathrm{DC}_{\kappa^{+}}$ scheme. Assume that in $U,|\mathscr{A}|=\kappa^{+}$and let $I_{6}=\left\{B \subseteq \mathscr{A}:|B|<\kappa^{+}\right\}$. By an argument as before, the $\mathrm{DC}_{\kappa^{+}}$-scheme fails in $U^{I_{6}}$. Every set of urelements in $U^{I_{6}}$ has tail cardinal $\kappa$, so Collection holds and the $\mathrm{DC}_{\kappa}$-scheme holds by Lemma 13

## 3. What is ZFC with urelements?

$\mathrm{ZFCU}_{\mathrm{R}}$ thus proves none of the axioms in the diagram of Theorem 7 . By contrast, $\mathrm{ZFCU}_{\mathrm{R}}+$ Collection yields many desirable consequences such as the $\mathrm{DC}_{\omega}$-scheme and the reflection principle. Moreover, Collection is also essential for applying standard constructions to models of ZFC with urelements. For example, let $U$ be a model of $\mathrm{ZFCU}_{\mathrm{R}}$ and $F, x \in U$ be such that $U \models(F$ is an ultrafilter on $x)$. One can form an internal ultrapower of $U$ as usual. Namely. for every $f, g \in U$ such that $U \models(f, g$ are functions on $x)$, define

$$
\begin{gathered}
f=_{F} g \text { if and only if } U \models(\{y \in x: f(y)=g(y)\} \in F) ; \\
{[f]=\left\{h \in U:(h \text { is a function on } x)^{U} \wedge h=_{F} f\right\} ;} \\
U / F=\left\{[h]: h \in U \wedge(h \text { is a function on } x)^{U}\right\} .
\end{gathered}
$$

For every $[f],[g] \in U / F$, define

$$
\begin{gathered}
{[g] \hat{\in}[f] \text { if and only if } U \models(\{y \in x: g(y) \in f(y)\} \in F) ;} \\
\hat{\mathscr{A}}([f]) \text { if and only if } U \models(\{y \in x: \mathscr{A}(f(y))\} \in F),
\end{gathered}
$$

Then the internal ultrapower is the model $\langle U / F, \hat{\in}, \hat{\mathscr{A}}\rangle$ (denoted by $U / F$ ). The Łoś theorem holds for $U / F$ if for every $\varphi$ and $\left[f_{1}\right], \ldots,\left[f_{n}\right] \in U / F$,

$$
U / F \models \varphi\left(\left[f_{1}\right], \ldots,\left[f_{n}\right]\right) \text { if and only if } U \models\left(\left\{y \in x: \varphi\left(f_{1}(y), \ldots, f_{n}(y)\right)\right\} \in F\right) .
$$

When $U \models$ ZFC, the Łoś theorem holds for all internal ultrapowers of $U$, which is commonly used in the study of large cardinals.

Theorem 19. Let $U$ be a model of $\mathrm{ZFCU}_{\mathrm{R}}$. The following are equivalent.
(1) The Łoś theorem holds for all internal ultrapowers of $U$.
(2) $U \models$ Collection.

Proof. The proof of (2) $\rightarrow$ (1) is standard, and the point here is that the use of Collection is essential.
For (1) $\rightarrow(2)$, suppose that Collection fails in $U$. Then by Theorem 7 , it follows that both Plenitude and Tail fail in $U$. In $U$, fix some $A \subseteq \mathscr{A}$ without a tail cardinal in $U$ and define $\kappa=\operatorname{Sup}\{\delta: \exists B \subseteq \mathscr{A}(|B|=$ $\delta \wedge B \cap A=\emptyset)\} . \kappa$ is an infinite limit cardinal in $U$, and in $U$ there is no set of urelements $D$ of size $\kappa$ that is disjoint from $A$. Let $F \in U$ be an ultrafilter on $\kappa$ containing all the unbounded subsets of $\kappa$. Suppose for reductio that the Łoś theorem holds for $U / F$. Let $i d$ be the identity function on $\kappa$ and $c_{A}$ be the constant function sending every $\alpha<\kappa$ to $A$. Since $U \models(\{\alpha<\kappa: \exists B \subseteq \mathscr{A}(|B|=\alpha \wedge B \cap A=\emptyset)\} \in F)$, by the Łoś theorem, $U / F \models \exists B \subseteq \mathscr{A}\left(|B|=[i d] \wedge B \cap\left[C_{A}\right]=\emptyset\right)$. Thus, there is some $g \in U$ such that

$$
U / F \models[g] \subseteq \mathscr{A} \wedge|[g]|=[i d] \wedge\left([g] \cap\left[C_{A}\right]=\emptyset\right)
$$

Let $x \in U$ be the set $\{\alpha<\kappa: g(\alpha) \subseteq \mathscr{A} \wedge|g(\alpha)|=|\alpha| \wedge(g(\alpha) \cap A=\emptyset)\}$. By the Łoś theorem again, $U \models x \in F$. In $U$, let $D=\bigcup_{\alpha \in x} g(\alpha)$, which is a set of urelements of size $\kappa$ that is disjoint from $A$ contradiction.

In the next section, we shall see that over $\mathrm{ZFCU}_{\mathrm{R}}$, Collection is also equivalent to the principle that every (properly defined) forcing relation is full. These results suggest that $\mathrm{ZU}+$ Collection +AC is a more robust theory than $\mathrm{ZFCU}_{\mathrm{R}}$. The following notation is thus justified, which has been adopted in $[8]{ }^{1}$
Definition 20. $\mathrm{ZFCU}=\mathrm{ZU}+$ Collection +AC .

## 4. Forcing over ZFCU $_{R}$

4.1. The standard approach. We now turn to forcing over countable transitive models of $\mathrm{ZFCU}_{\mathrm{R}}$. With urelements, a standard definition of $\mathbb{P}$-names for a given forcing poset $\mathbb{P}$ is to treat each urelement as its own name. This approach has been adopted in all existing studies such as [1], [6] and [7].
Definition 21. Let $\mathbb{P}$ be a forcing poset. $\dot{x}$ is a $\mathbb{P}$-name ${ }_{\#}$ if and only if either $\dot{x}$ is a urelement, or $\dot{x}$ is a set of ordered-pairs $\langle\dot{y}, p\rangle$, where $\dot{y}$ is a $\mathbb{P}$-name $\#$ and $p \in \mathbb{P} . U_{\#}^{\mathbb{P}}=\left\{\dot{x}: \dot{x}\right.$ is a $\mathbb{P}$-name $\left.{ }_{\#}\right\}$.
Definition 22. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}, \mathbb{P} \in M$ be a forcing poset, and $G$ be an $M$-generic filter over.
(1) $M_{\#}^{\mathbb{P}}=M \cap U_{\#}^{\mathbb{P}}$.
(2) For every $\dot{x} \in M_{\#}^{\mathbb{P}}$,

$$
\dot{x}_{G}= \begin{cases}\dot{x} & \text { if } \mathscr{A}(x) \\ \left\{\dot{y}_{G}: \exists p \in G\langle\dot{y}, p\rangle \in \dot{x}\right\} & \text { otherwise }\end{cases}
$$

(3) $M[G]_{\#}=\left\{\dot{x}_{G}: \dot{x} \in M_{\#}^{\mathbb{P}}\right\}$.
(4) For every $\dot{x}_{1}, \ldots, \dot{x}_{n} \in M_{\#}^{\mathbb{P}}$ and $p \in \mathbb{P}, p \Vdash_{\#} \varphi\left(\dot{x}_{1}, \ldots, \dot{x}_{n}\right)$ if and only if for every $M$-generic $H$ such that $p \in H, M[H] \models \varphi\left(\dot{x}_{1 H}, \ldots, \dot{x}_{n H}\right)$.
With these definitions, one can easily prove the forcing theorems for $\Vdash_{\#}$ by making trivial adjustments to the standard argument. And it is clear that $M[G]_{\#}$ is transitive, $M \subseteq M[G]_{\#}$, and $G \in M[G]_{\#}$. In fact, $M[G]_{\#}$ is a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}$ (see Appendix).

However, an important feature of forcing is missing in this approach, which is why the subscript \# is added. Given $M$ and $\mathbb{P}$ as above, the forcing relation $\Vdash_{\#}$ given by $\mathbb{P}$ is said to be full if whenever $p \Vdash_{\#}$ $\exists y \varphi\left(y, \dot{x}_{1}, \ldots, \dot{x}_{n}\right)$ for $\dot{x}_{1}, \ldots, \dot{x}_{n} \in M_{\#}^{\mathbb{P}}$, there is a $\dot{y} \in M_{\#}^{\mathbb{P}}$ such that $p \Vdash_{\#} \varphi\left(\dot{y}, \dot{x}_{1}, \ldots, \dot{x}_{n}\right)$. It is a standard result that if $M \models$ ZFC, then for every forcing poset in $M$, its forcing relation is full. Fullness has various applications. e.g., it is used for iterated forcing and Boolean-valued ultrapowers.

Remark 23. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}$ with urelements. Then for every $\mathbb{P}$ with a maximal antichain with at least two elements, its forcing relation $\Vdash_{\#}$ is not full.
Proof. Suppose that $\mathbb{P} \in M$ has a maximal antichain $\left\langle p_{i}: i \in I\right\rangle$ indexed by some $I(|I|>1)$. Let $\left\langle a_{i}: i \in I\right\rangle$ be some urelements such that at least two of them are distinct. Consider the $\mathbb{P}$-name $\dot{\#} \dot{x}=\left\{\left\langle a_{i}, p_{i}\right\rangle: i \in I\right\}$. It follows that $1_{\mathbb{P}} \Vdash_{\#} \exists y(y \in \dot{x})$. But if $1_{\mathbb{P}} \Vdash_{\#} \dot{y} \in \dot{x}$ for some $\dot{y} \in M_{\#}^{\mathbb{P}}$, then $\dot{y}$ must be some $a_{i}$, which is impossible since one can take an $M$-generic filter containing $p_{j}$, where $a_{j} \neq a_{i}$.

A diagnosis is that $M_{\#}^{\mathbb{P}}$ contains too few names. In pure set theory, whenever $f$ is a function from an antichain in a forcing poset $\mathbb{P}$ to some $\mathbb{P}$-names, we can define a mixture of $f, \dot{y}$, such that $p \Vdash f(p)=\dot{y}$ for every $p \in \operatorname{dom}(f)$. But as we have seen, $M_{\#}^{\mathbb{P}}$ does not even contain a mixture of two urelements, which motivates a new definition of $\mathbb{P}$-names with urelements.

[^0]
### 4.2. A new approach.

Definition 24. Let $\mathbb{P}$ be a forcing poset. $\dot{x}$ is a $\mathbb{P}$-name if and only if (i) $\dot{x}$ is a set of ordered-pairs $\langle y, p\rangle$ where $p \in \mathbb{P}$ and $y$ is either a $\mathbb{P}$-name or a urelement, and (ii) whenever $\langle a, p\rangle,\langle y, q\rangle \in \dot{x}$, where $a$ is a urelement and $a \neq y, p$ and $q$ are incompatible. $U^{\mathbb{P}}=\{\dot{x}: \dot{x}$ is a $\mathbb{P}$-name $\}$.
Some key differences between $U^{\mathbb{P}}$ and $U_{\#}^{\mathbb{P}}$ should be noted. First, no urelement is a $\mathbb{P}$-name in $U^{\mathbb{P}}$, and each urelement $a$ is represented by $\{\langle a, 1\rangle\}$ rather than itself. Second, when $\langle a, p\rangle \in \dot{x}$ for some urelement $a$, this indicates that $a$ will be identical to, rather than is a member of, $\dot{x}_{G}$ for any generic filter $G$ containing $p$.
Definition 25. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}, \mathbb{P} \in M$ be a forcing poset and $G$ be an $M$-generic filter over $\mathbb{P}$.
(1) $M^{\mathbb{P}}=U^{\mathbb{P}} \cap M$
(2) For every $\dot{x} \in M^{\mathbb{P}}$,

$$
\dot{x}_{G}= \begin{cases}a & \text { if } \mathscr{A}(a) \text { and }\langle a, p\rangle \in \dot{x} \text { for some } p \in G \\ \left\{\dot{y}_{G}:\langle\dot{y}, p\rangle \in \dot{x} \text { for some } \dot{y} \in M^{\mathbb{P}} \text { and } p \in G\right\} & \text { otherwise }\end{cases}
$$

(3) $M[G]=\left\{\dot{x}_{G}: \dot{x} \in M^{\mathbb{P}}\right\}$.
(4) For every urelement $a \in M, \check{a}=\left\{\left\langle a, 1_{\mathbb{P}}\right\rangle\right\}$; for every set $x \in M, \check{x}=\left\{\left\langle\check{y}, 1_{\mathbb{P}}\right\rangle: y \in x\right\}$.
(5) For every $\dot{x}_{1}, \ldots, \dot{x}_{n} \in M^{\mathbb{P}}$ and $p \in \mathbb{P}, p \Vdash \varphi\left(\dot{x}_{1}, \ldots, \dot{x}_{n}\right)$ if and only if for every $M$-generic $H$ such that $p \in H, M[H] \models \varphi\left(\dot{x}_{1 H}, \ldots, \dot{x}_{n H}\right)$.
Note that $\dot{x}_{G}$ is well-defined given the incompatibility condition (ii) in Definition 24. It is shown in the Appendix that $M[G]$ is in fact the same as $M[G]_{\#}$.

Lemma 26. Let $M$ be a countable transitive model of $M, \mathbb{P} \in M$ be a forcing poset, and $G$ be an $M$-generic filter over $\mathbb{P}$. Then
(1) $M \subseteq M[G]$;
(2) $G \in M[G]$;
(3) $M[G]$ is transitive;
(4) $O r d \cap M=O r d \cap M[G]$;
(5) For every transitive model $N$ of $\mathrm{ZFCU}_{\mathrm{R}}$ such that $G \in N$ and $M \subseteq N, M[G] \subseteq N$;
(6) $\mathscr{A} \cap M=\mathscr{A} \cap M[G]$.

Proof. (1)-(5) are all proved by standard text-book arguments as in [11, Ch.VII]. (6) is clear by the construction of $M[G]$ because every urelement in $M[G]$ must come from $\operatorname{ker}(\dot{x})$ for some $\dot{x} \in M^{\mathbb{P}}$.
Lemma 27. $\operatorname{ker}\left(\dot{x}_{G}\right) \subseteq \operatorname{ker}(\dot{x})$, for every $\dot{x} \in M^{\mathbb{P}}$. Hence, every set of urelements in $M[G]$ is a subset of some set of urelements in $M$.

Proof. By induction on the rank of $\dot{x}$, and we may assume that $\dot{x}_{G}$ is a set. Since $\operatorname{ker}\left(\dot{x}_{G}\right) \subseteq \bigcup\left\{\operatorname{ker}\left(\dot{y}_{G}\right): \dot{y} \in\right.$ $\operatorname{dom}(\dot{x})\}$ and by the induction hypothesis we have $\operatorname{ker}\left(\dot{y}_{G}\right) \subseteq \operatorname{ker}(\dot{y}) \subseteq \operatorname{ker}(\dot{x})$ for every $\dot{y} \in \operatorname{dom}(\dot{x})$, so the lemma follows.

Next we need to prove the forcing theorems for $\Vdash$, i.e., " $p \Vdash \varphi$ " is definable in $M$ for every $\varphi$ and every truth in a generic extension is forced by some condition in the corresponding generic filter. The first step is to define an internal forcing relation.
Definition 28. Let $M$ and $\mathbb{P}$ be as before. The forcing language $\mathscr{L}_{\mathbb{P}}^{M}$ contains $\{\subseteq,=, \in, \mathscr{A}, \stackrel{\mathscr{A}}{=}\}$ as the nonlogical symbols and every $\mathbb{P}$-name in $M^{\mathbb{P}}$ as a constant symbol. For every $p \in \mathbb{P}$ and $\varphi \in \mathscr{L}_{\mathbb{P}}^{M}$, we define $p \Vdash^{*} \varphi$ by recursion as follows.
(1) $p \Vdash^{*} \mathscr{A}\left(\dot{x}_{1}\right)$ if and only if $\left\{q \in \mathbb{P}: \exists\langle a, r\rangle \in \dot{x}_{1}(\mathscr{A}(a) \wedge q \leq r)\right\}$ is dense below $p$.
(2) $p \Vdash^{*} \dot{x}_{1} \stackrel{\mathscr{A}}{=} \dot{x}_{2}$ if and only if $\left\{q \in \mathbb{P}: \exists a, r_{1}, r_{2}\left(\mathscr{A}(a) \wedge\left\langle a, r_{1}\right\rangle \in \dot{x}_{1} \wedge\left\langle a, r_{2}\right\rangle \in \dot{x}_{2} \wedge q \leq r_{1}, r_{2}\right)\right\} \cup\{q \in$ $\left.\mathbb{P}: \forall\left\langle a_{1}, r_{1}\right\rangle \in \dot{x}_{1}\left(\mathscr{A}\left(a_{1}\right) \rightarrow q \perp r_{1}\right) \wedge \forall\left\langle a_{2}, r_{2}\right\rangle \in \dot{x}_{2}\left(\mathscr{A}\left(a_{2}\right) \rightarrow q \perp r_{2}\right)\right\}$ is dense below $p$.
(3) $p \Vdash^{*} \dot{x}_{1} \in \dot{x}_{2}$ if and only if $\left\{q \in \mathbb{P}: \exists\langle\dot{y}, r\rangle \in \dot{x}_{2}\left(q \leq r \wedge \dot{y} \in M^{\mathbb{P}} \wedge q \Vdash^{*} \dot{y}=\dot{x}_{1}\right)\right\}$ is dense below $p$.
(4) $p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2}$ if and only if for every $\dot{y} \in M^{\mathbb{P}}$ and $r, q \in \mathbb{P}$, if $\langle\dot{y}, r\rangle \in \dot{x}_{1}$ and $q \leq p, r$, then $q \Vdash^{*} \dot{y} \in \dot{x}_{2}$.
(5) $p \Vdash^{*} \dot{x}_{1}=\dot{x}_{2}$ if and only if $p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2}, p \Vdash^{*} \dot{x}_{2} \subseteq \dot{x}_{1}$ and $p \Vdash^{*} \dot{x}_{1} \xlongequal{\mathscr{A}} \dot{x}_{2}$.
(6) $p \vdash^{*} \neg \varphi$ if and only if there is no $q \leq p$ such that $q \Vdash^{*} \varphi$.
(7) $p \Vdash^{*} \varphi \wedge \psi$ if and only if $p \Vdash^{*} \varphi$ and $p \Vdash^{*} \psi$.
(8) $p \Vdash^{*} \exists x \varphi$ if and only if $\left\{q \in \mathbb{P}\right.$ : there is some $\dot{z} \in M^{\mathbb{P}}$ such that $\left.q \Vdash^{*} \varphi(\dot{z})\right\}$ is dense below $p$.

Lemma 29. Let $M$ and $\mathbb{P}$ be as before. For every $p, q \in \mathbb{P}$,
(1) If $p \Vdash^{*} \varphi$ and $q \leq p$, then $q \Vdash^{*} \varphi$.
(2) If $\left\{r \in \mathbb{P}: r \Vdash^{*} \varphi\right\}$ is dense below $p, p \Vdash^{*} \varphi$.

Lemma 30. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}, \mathbb{P} \in M$ be a forcing poset and $G$ be an $M$-generic filter over $\mathbb{P}$. For every $\dot{x}_{1}, \ldots, \dot{x}_{n} \in M^{\mathbb{P}}$,
(1) If $p \in G$ and $p \vdash^{*} \varphi\left(\dot{x}_{1}, \ldots, \dot{x}_{n}\right)$, then $M[G] \models \varphi\left(\dot{x}_{1_{G}}, \ldots, \dot{x}_{n_{G}}\right)$.
(2) If $M[G] \models \varphi\left(\dot{x}_{1_{G}}, \ldots, \dot{x}_{n_{G}}\right)$, then there is some $p \in G\left(p \Vdash^{*} \varphi\left(\dot{x}_{1}, \ldots, \dot{x}_{n}\right)\right.$

Proof. Since the Boolean and quantifier cases can be proved in the same way as in [11, Chapter VII. Theorem 3.5], we omit their proofs. It remains to show that the lemma holds for all atomic formulas, which we shall prove by induction on the rank of the $\mathbb{P}$-names.

Case 1. $\varphi$ is $\dot{x}_{1} \in \dot{x}_{2}$. The argument is the same as in [11, Chapter VII, Theorem 3.5].
Case 2. $\varphi$ is $\mathscr{A}(\dot{x})$. For (2), suppose that $\dot{x}_{G}$ is some urelement $b$. Then $\langle b, p\rangle \in \dot{x}$ for some $p \in G$, so $\{q \in \mathbb{P}: \exists\langle a, r\rangle \in \dot{x}(\mathscr{A}(a) \wedge q \leq r)\}$ is dense below $p$ and hence $p \Vdash^{*} \mathscr{A}(\dot{x})$. For (1), suppose that $p \Vdash^{*} \mathscr{A}(\dot{x})$ for some $p \in G$. Then there is some $q \in G$ such that $\langle b, r\rangle \in \dot{x}$ for some $r \geq q$ and urelement $b$. Thus, $\dot{x}_{G}=b$.

Case 3. $\varphi$ is $\dot{x}_{1}=\dot{x}_{2}$. For (2), suppose that $\dot{x}_{1_{G}}=\dot{x}_{2_{G}}$.
Subcase 3.1. $\dot{x}_{1_{G}}=\dot{x}_{2_{G}}=b$ for some urelement $b$. Then $\left\langle b, s_{1}\right\rangle \in \dot{x}_{1}$ and $\left\langle b, s_{2}\right\rangle \in \dot{x}_{2}$ for some $s_{1}, s_{2} \in G$. Fix some $p \in G$ such that $p \leq s_{1}, s_{2}$. Observe first that $p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2}$ and $p \Vdash^{*} \dot{x}_{2} \subseteq \dot{x}_{1}$ trivially hold: for any $\mathbb{P}$-name $\dot{y}$ and $r \in \mathbb{P}$ such that $\langle\dot{y}, r\rangle \in \dot{x}_{1}$ (or $\dot{x}_{2}$ ), $p$ must be incompatible with $r$ because $r$ is incompatible with $s_{1}\left(\right.$ or $\left.s_{2}\right)$. Moreover, $p \Vdash^{*} \dot{x}_{1} \stackrel{\mathscr{A}}{=} \dot{x}_{2}$ because $\left\{q \in \mathbb{P}: \exists a, r_{1}, r_{2}\left(\mathscr{A}(a) \wedge\left\langle a, r_{1}\right\rangle \in \dot{x}_{1} \wedge\left\langle a, r_{2}\right\rangle \in \dot{x}_{2} \wedge q \leq r_{1}, r_{2}\right)\right\}$ is clearly dense below $p$. Hence, $p \Vdash^{*} \dot{x}_{1}=\dot{x}_{2}$.

Subcase 3.2. $\dot{x}_{1_{G}}$ is a set. We first use a standard text-book argument to show that $p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2}$ and $p \Vdash^{*} \dot{x}_{2} \subseteq \dot{x}_{1}$ for some $p \in G$. Define:

$$
\begin{aligned}
& D_{1}=\left\{p \in \mathbb{P}: p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2} \wedge p \Vdash^{*} \dot{x}_{2} \subseteq \dot{x}_{1}\right\} ; \\
& D_{2}=\left\{p \in \mathbb{P}: \exists\left\langle\dot{y}_{1}, q_{1}\right\rangle \in \dot{x}_{1}\left(p \leq q_{1} \wedge \forall\left\langle\dot{y}_{2}, q_{2}\right\rangle \in \dot{x}_{2} \forall r \leq q_{2}\left(r \Vdash^{*} \dot{y}_{1}=\dot{y}_{2} \rightarrow p \perp r\right)\right)\right\} ; \\
& D_{3}=\left\{p \in \mathbb{P}: \exists\left\langle\dot{y}_{2}, q_{2}\right\rangle \in \dot{x}_{2}\left(p \leq q_{2} \wedge \forall\left\langle\dot{y}_{1}, q_{1}\right\rangle \in \dot{x}_{1} \forall r \leq q_{1}\left(r \vdash^{*} \dot{y}_{2}=\dot{y}_{1} \rightarrow p \perp r\right)\right)\right\} .
\end{aligned}
$$

If $p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2}$, then there are $\left\langle\dot{y}_{1}, q_{1}\right\rangle \in \dot{x}_{1}$ and $r \leq p, q_{1}$ such that $r \nVdash^{*} \dot{y}_{1} \in \dot{x}_{2}$; so there is an $s \leq r$ such that for every $\left\langle\dot{y}_{2}, q_{2}\right\rangle \in \dot{x}_{2}$ and $s^{\prime} \leq q_{2}$. If $s^{\prime} \Vdash^{*} \dot{y}_{1}=\dot{y}_{2}$, then $s \perp s^{\prime}$. Hence, $s \leq p$ and $s \in D_{2}$. Similarly, if $p \Vdash^{*} \dot{x}_{2} \subseteq \dot{x}_{1}$, then $p$ will have an extension in $D_{3}$. This shows that $D_{1} \cup D_{2} \cup D_{3}$ is dense. However,
$G \cup\left(D_{2} \cup D_{3}\right)$ must be empty. Suppose for reductio that $p \in G \cap D_{2}$. Fix some $\left\langle\dot{y}_{1}, q_{1}\right\rangle \in \dot{x}_{1}$ with $p \leq q_{1}$ that witnesses $p \in D_{2}$. It follows that $\dot{y}_{1 G}=\dot{y}_{2 G}$ for some $\left\langle\dot{y}_{2}, q_{2}\right\rangle \in \dot{x}_{2}$ with $q_{2} \in G$. By the induction hypothesis, there is some $r \in G$ such that $r \leq q_{2}$ and $r \Vdash^{*} \dot{y}_{1}=\dot{y}_{2}$. But $p$ must be incompatible with such $r$, which is a contradiction. The same argument shows that $G \cap D_{3}$ is empty. Therefore, there is some $p \in G$ such that $p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2}$ and $p \Vdash^{*} \dot{x}_{2} \subseteq \dot{x}_{1}$.

Now we wish to find some $q \in G$ such that $q \Vdash^{*} \dot{x}_{1} \stackrel{\mathscr{L}}{=} \dot{x}_{2}$. Define:

$$
\begin{aligned}
& E_{1}=\left\{q \in \mathbb{P}: \forall r \leq q\left[\forall\left\langle a_{1}, s_{1}\right\rangle \in \dot{x}_{1}\left(\mathscr{A}(a) \rightarrow r \perp s_{1}\right) \wedge \forall\left\langle a_{2}, s_{2}\right\rangle \in \dot{x}_{2}\left(\mathscr{A}\left(a_{2}\right) \rightarrow r \perp s_{2}\right)\right]\right\} \\
& \left.E_{2}=\left\{q \in \mathbb{P}: \exists\langle a, r\rangle \in \dot{x}_{1}(\mathscr{A}(a) \wedge q \leq r)\right)\right\} \\
& E_{3}=\left\{q \in \mathbb{P}: \exists\langle a, r\rangle \in \dot{x}_{2}(\mathscr{A}(a) \wedge q \leq r)\right\}
\end{aligned}
$$

$E_{1} \cup E_{2} \cup E_{3}$ is dense. But if there is some $q \in G \cap\left(E_{2} \cup E_{3}\right)$, either $\dot{x}_{1_{G}}$ or $\dot{x}_{2_{G}}$ would be a urelement. Thus there is some $q \in G \cap E_{1}$ such that the set

$$
\left\{r \in \mathbb{P}: \forall\left\langle a_{1}, s_{1}\right\rangle \in \dot{x}_{1}\left(\mathscr{A}\left(a_{1}\right) \rightarrow r \perp s_{1}\right) \wedge \forall\left\langle a_{2}, s_{2}\right\rangle \in \dot{x}_{2}\left(\mathscr{A}\left(a_{2}\right) \rightarrow r \perp s_{2}\right)\right\}
$$

is dense below $q$. Therefore, $q \Vdash^{*} \dot{x}_{1} \stackrel{\mathscr{A}}{=} \dot{x}_{2}$. A common extension of $p$ and $q$ in $G$ will then force $\dot{x}_{1}=\dot{x}_{2}$.
To show that (1) holds for Case 3, suppose that for some $p \in G, p \Vdash^{*} \dot{x}_{1}=\dot{x}_{2}$.
Subcase 3.3. $\dot{x}_{1_{G}}=b$ for some urelement $b$. Then $\langle b, r\rangle \in \dot{x}_{1}$ for some $r \in G$. Define:

$$
\begin{aligned}
& F_{1}=\left\{q \in \mathbb{P}: \exists a, s_{1}, s_{2}\left(\mathscr{A}(a) \wedge\left\langle a, s_{1}\right\rangle \in \dot{x}_{1} \wedge\left\langle a, s_{2}\right\rangle \in \dot{x}_{2} \wedge q \leq s_{1}, s_{2}\right)\right\} . \\
& F_{2}=\left\{q \in \mathbb{P}: \forall\left\langle a, s_{1}\right\rangle \in \dot{x}_{1}\left(\mathscr{A}(a) \rightarrow q \perp s_{1}\right) \wedge \forall\left\langle a, s_{2}\right\rangle \in \dot{x}_{2}\left(\mathscr{A}(a) \rightarrow q \perp s_{2}\right)\right\} .
\end{aligned}
$$

Since $p \Vdash^{*} \dot{x}_{1} \stackrel{\mathscr{A}}{=} \dot{x}_{2}, F_{1} \cup F_{2}$ is dense below $p$. But clearly $F_{2} \cap G$ is empty as $\langle b, r\rangle \in \dot{x}_{1}$, so there is some $q \in F_{1} \cap G$. It follows that $\left\langle b, s_{1}\right\rangle \in \dot{x}_{1}$ and $\left\langle b, s_{2}\right\rangle \in \dot{x}_{2}$ for some $s_{1}, s_{2} \in G$. Therefore, $\dot{x}_{2_{G}}=b=\dot{x}_{2_{G}}$.

Subcase 3.4. $\dot{x}_{1_{G}}$ is a set. Suppose for reductio that $\dot{x}_{2_{G}}$ is some urelement $b$ and so $\langle b, r\rangle \in \dot{x}_{2}$ for some $r \in G$. Since $p \Vdash^{*} \dot{x}_{1} \stackrel{\mathscr{A}}{=} \dot{x}_{2}$, it follows that there are some urelement $a$ and $s \in G$ such that $\langle a, s\rangle \in \dot{x}_{1}$. This implies that $\dot{x}_{1_{G}}=a$, which is a contradiction. Hence, $\dot{x}_{2_{G}}$ is a set, so it remains to show that $\dot{x}_{1_{G}}$ and $\dot{x}_{2_{G}}$ have the same members. If $\dot{y}_{G} \in \dot{x}_{1_{G}}$, then $\langle\dot{y}, r\rangle \in \dot{x}_{1}$ for some $r \in G$. So there is some $q \in G$ with $q \leq p, r$, and $q \Vdash^{*} \dot{y} \in \dot{x}_{2}$ because $p \Vdash^{*} \dot{x}_{1} \subseteq \dot{x}_{2}$. By the induction hypothesis, $\dot{y}_{G} \in \dot{x}_{2_{G}}$. The same argument will show that $\dot{x}_{2_{G}} \subseteq \dot{x}_{1_{G}}$.

By standard arguments, this lemma yields the forcing theorem with urelements.
Theorem 31 (The Forcing Theorem with Urelements $\Vdash$ ). Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}$ and $\mathbb{P} \in M$ be a forcing poset. Then for every $\dot{x}_{1}, \ldots, \dot{x}_{n} \in M^{\mathbb{P}}$,
(1) $p \Vdash^{*} \varphi\left(\dot{x}_{1}, \ldots, \dot{x}_{n}\right)$ if and only if $p \Vdash \varphi\left(\dot{x}_{1}, \ldots, \dot{x}_{n}\right)$.
(2) For every $M$-generic filter $G$ over $\mathbb{P}, M[G] \models \varphi\left(\dot{x}_{1_{G}}, \ldots, \dot{x}_{n_{G}}\right)$ if and only if $\exists p \in G\left(p \Vdash \varphi\left(\dot{x}_{1}, \ldots, \dot{x}_{n}\right)\right.$.
4.3. Fullness is equivalent to Collection. We first verify that $M^{\mathbb{P}}$ is now closed under mixtures.

Lemma 32. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}$ and $\mathbb{P} \in M$ be a forcing poset. For every function $f: \operatorname{dom}(f) \rightarrow M^{\mathbb{P}}$ in $M$, where $\operatorname{dom}(f)$ is an antichain in $\mathbb{P}$, there is a $\dot{v} \in M^{\mathbb{P}}$ such that $p \Vdash \dot{v}=f(p)$ for every $p \in \operatorname{dom}(f)$.
Proof. In $M$, we define $\dot{v}$ as follows.

$$
\dot{v}=\bigcup_{p \in \operatorname{dom}(f)}\{\langle y, r\rangle \in \operatorname{dom}(f(p)) \times \mathbb{P}: \exists q(\langle y, q\rangle \in f(p) \wedge r \leq p, q)\} .
$$

We first check that $\dot{v}$ satisfies the incompatbility condition (ii) in Definition 24 Consider any $\left\langle a, r_{1}\right\rangle \in \dot{v}$ for some urelement $a$. Then there are $p_{1}, q_{1}$ such that $p_{1} \in \operatorname{dom}(f)$ and $\left\langle a, q_{1}\right\rangle \in f\left(p_{1}\right)$ and $r_{1} \leq p_{1}, q_{1}$. For any $\left\langle x, r_{2}\right\rangle \in \dot{v}$ with $x \neq a$, there are $p_{2}, q_{2}$ such that $p_{2} \in \operatorname{dom}(f)$ and $\left\langle x, q_{2}\right\rangle \in f\left(p_{2}\right)$ and $r_{2} \leq p_{2}, q_{2}$. If $p_{1}=p_{2}$, then $r_{1}$ is incompatible with $r_{2}$ because $f\left(p_{1}\right)$ is a $\mathbb{P}$-name. If not, $r_{1}$ is incompatible with $r_{2}$ because $\operatorname{dom}(f)$ is an antichain.

Fix a $p \in \operatorname{dom}(f)$. We show that $p \Vdash \dot{v}=f(p)$. Let $G$ be an $M$-generic filter over $\mathbb{P}$ that contains $p$.
Case 1. $\dot{v}_{G}$ is some urelement $a$. Then $\langle a, r\rangle \in \dot{v}$ for some $r \in G$. So for some $p^{\prime} \in \operatorname{dom}(f)$ and $q$, $\langle a, q\rangle \in f\left(p^{\prime}\right)$ and $r \leq p^{\prime}, q$. So $p^{\prime}, q \in G$ and $p^{\prime}=p$. Therefore, $\dot{v}_{G}=f(p)_{G}$.

Case 2. $\dot{v}_{G}$ is a set. Then $f(p)_{G}$ must be a set. Otherwise, $f(p)_{G}$ is some urelement $a$ and there will be some $q \in G$ such that $\langle a, q\rangle \in f(p)$; then there is some $s \in G$ such that $s \leq q, p$ so $\langle a, s\rangle \in \dot{v}$, which means $\dot{v}_{G}$ is a urelement-contradiction. For every $\dot{x}_{G} \in \dot{v}_{G}$ with $\langle\dot{x}, r\rangle \in \dot{v}$ and $r \in G,\langle\dot{x}, q\rangle \in f\left(p^{\prime}\right)$ and $r \leq p^{\prime}, q$ for some $p^{\prime} \in \operatorname{dom}(f)$ and $q$; so $p^{\prime}=p$ and $\dot{x}_{G} \in f(p)_{G}$. This shows that $\dot{v}_{G} \subseteq f(p)_{G}$. Consider any $\dot{x}_{G} \in f(p)_{G}$ such that $\langle\dot{x}, q\rangle \in f(p)$ for some $q \in G$. Let $r \in G$ be a common extension of $p$ and $q$. It follows that $\langle\dot{x}, r\rangle \in \dot{v}$ and so $\dot{x}_{G} \in \dot{v}_{G}$. This shows that $f(p)_{G} \subseteq \dot{v}_{G}$ and the proof is completed.

Theorem 33. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{R}$. The following are equivalent.
(1) $M \models$ Collection.
(2) For every forcing notion $\mathbb{P} \in M$, its forcing relation $\Vdash$ is full.

Proof. (1) $\rightarrow$ (2). Fix some $\mathbb{P} \in M$ and suppose that $p \Vdash \exists y \varphi(y)$ for some $p \in \mathbb{P}$. In $M$, by AC we can fix a maximal antichain $X$ in the subposet $\mathbb{Q}=\left\{q \in \mathbb{P}: q \leq p \wedge \exists \dot{y} \in M^{\mathbb{P}} q \Vdash \varphi(\dot{y})\right\}$. By Collection and AC in $M$, we can pick a $\dot{y}_{q} \in M^{\mathbb{P}}$ such that $q \Vdash \varphi\left(\dot{y}_{q}\right)$ for every $q \in X$. By lemma 32 , there is a $\dot{v} \in M^{\mathbb{P}}$ such that $q \Vdash \dot{y}_{q}=\dot{v}$ for every $q \in X$. Suppose that $p \nVdash \varphi(\dot{v})$ for reductio. Then there will be some $r \in Q$ such that $r \Vdash \neg \varphi(\dot{v})$, which means $r$ is incompatible with every $q \in X$, but this contradicts the maximality of $X$.
(2) $\rightarrow$ (1). Suppose that $M \models \forall x \in w \exists y \varphi(x, y, u)$. Let $\mathbb{P}$ be the forcing poset $w \cup\{w\}$ such that for every $p, q \in$ $\mathbb{P}, p \leq q$ if and only if $p=q$ or $q=w$. That is, $w$ is $1_{\mathbb{P}}$, while the members of $w$ constitute the only maximal antichain. Thus, $M[G]=M$ for every generic filter $G$ over $\mathbb{P}$. Define $\dot{x} \in M^{\mathbb{P}}$ to be $\{\langle\check{z}, x\rangle: z \in x \wedge x \in w\}$. For every generic filter $G$ over $\mathbb{P}$, since $\dot{x}_{G}=x$ for the unique $x \in G \cap w$, it follows that $M[G] \models \exists y \varphi\left(\dot{x}_{G}, y, u\right)$. Thus, $1_{\mathbb{P}} \Vdash \exists y \varphi(\dot{x}, y, \check{u})$ and by (2), $1_{\mathbb{P}} \Vdash \varphi(\dot{x}, \dot{y}, \check{u})$ for some $\dot{y} \in M^{\mathbb{P}}$. For every $x \in w$, let $G$ be the filter containing $x$. Then $M[G] \models \varphi\left(x, \dot{y}_{G}, u\right)$; so $M \models \varphi\left(x, \dot{y}_{G}, u\right)$ and $\operatorname{ker}\left(\dot{y}_{G}\right) \subseteq \operatorname{ker}(\dot{y})$ by Lemma 27. It follows that $M \models \forall x \in w \exists y \in V(\operatorname{ker}(\dot{y})) \varphi(x, y, u)$, which suffices for Collection by Proposition 2 .

### 4.4. The fundamental theorem.

Lemma 34. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}, \mathbb{P} \in M$ be a forcing poset, and $G$ be an $M$-generic filter over $\mathbb{P}$. Then
(1) $M[G]$ is a countable transitive model of ZCU ;
(2) $M[G] \mid=$ Collection if $M \models$ Collection.

Proof. The proof of (1) is a standard text-book argument as in Kunen [11, Ch.VII]. For (2), suppose that $M[G] \vDash \forall v \in \dot{w}_{G} \exists y \varphi\left(v, y, \dot{u}_{G}\right)$ for some $\dot{w}_{G}$ and $\dot{u}_{G}$. In $M$, define

$$
x=\left\{\langle\dot{x}, p\rangle \in\left(\operatorname{dom}(\dot{w}) \cap M^{\mathbb{P}}\right) \times \mathbb{P}: \exists \dot{y} \in M^{\mathbb{P}} p \Vdash \varphi(\dot{x}, \dot{y}, \dot{u})\right\} .
$$

By Collection in $M$, there is a set of $\mathbb{P}$-names $v$ such that for every $\langle\dot{x}, p\rangle \in x$, there is a $\dot{y} \in v$ with $p \Vdash$ $\varphi(\dot{x}, \dot{y}, \dot{u})$. Define $\dot{v}$ to be $v \times\left\{1_{\mathbb{P}}\right\}$. It's routine to check that $M[G] \models \forall x \in \dot{w}_{G} \exists y \in \dot{v}_{G} \varphi\left(x, y, \dot{u}_{G}\right)$.

A more difficult question is whether forcing preserves Replacement when the ground model $M$ does not satisfy Collection. When $M$ is a model of ZF, the standard argument for $M[G] \models$ Replacement appeals to

Collection in $M$. But this move is not allowed when $M$ only satisfies $\mathrm{ZFCU}_{\mathrm{R}}$. A new argument is thus needed.

Definition 35. Let $M$ and $\mathbb{P}$ be as before and $A \in M$ be a set of urelements. For every urelement $a \in M$, let $\stackrel{A}{a}=a$. For every $\dot{x} \in M^{\mathbb{P}}$, we define $\stackrel{A}{\dot{x}}$ (the A-purification of $\dot{x}$ ) as follows.

$$
\stackrel{A}{\dot{x}}=\left\{\langle\hat{y}, p\rangle:\langle y, p\rangle \in \dot{x} \wedge\left(y \in M^{\mathbb{P}} \vee y \in A\right)\right\} .
$$

That is, $\stackrel{A}{\dot{x}}$ is obtained by hereditarily throwing out the urelements used to build $\dot{x}$ that are not in $A$.
Proposition 36. Let $A \in M$ be a set of urelements such that $\operatorname{ker}(\mathbb{P}) \subseteq A$. For every $\dot{x} \in M^{\mathbb{P}}, \stackrel{A}{\dot{x}} \in M^{\mathbb{P}}$ and $\operatorname{ker}(\stackrel{A}{\dot{x}}) \subseteq A$.

Proof. By induction on the rank of $\dot{x}$. To show that $\stackrel{A}{\dot{x}}$ is always a $\mathbb{P}$-name, we only need to check the incompatibility condition in Defnition 24 holds. Suppose that $\langle a, p\rangle,\langle y, q\rangle \in \stackrel{A}{\dot{x}}$, where $a$ is a urelement and $y \neq a$. If $y$ is another urelement in $\operatorname{dom}(\dot{x})$, then $p$ and $q$ are incompatible; otherwise $y$ is some $\dot{z}$, where $\langle\dot{z}, q\rangle \in \dot{x}$ and $\dot{z}$ is a $\mathbb{P}$-name, then $p$ and $q$ are incompatible because no urelement is a $\mathbb{P}$-name. $\operatorname{ker}(\dot{x}) \subseteq A$ because $\operatorname{ker}(\stackrel{A}{\dot{x}})$ is contained in $\bigcup_{y \in \operatorname{dom}(\dot{x})} \operatorname{ker}(\stackrel{A}{y}) \cup \operatorname{ker}(\mathbb{P})$, which is a subset of $A$ by the induction hypothesis.

Theorem 37. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}, \mathbb{P} \in M$ be a forcing poset and $G$ be $M$ generic over $\mathbb{P}$. Then $M[G]=$ Replacement.

Proof. Suppose that $M[G] \models \forall v \in \dot{w}_{G} \exists!y \varphi\left(v, y, \dot{u}_{G}\right)$. Let $A=\operatorname{ker}(\dot{w}) \cup \operatorname{ker}(\mathbb{P}) \cup \operatorname{ker}(\dot{u})$. By Lemma 34. we may assume $M$ does not satisfy Collection and hence has a proper class of urelements.
Lemma 38. For every $\dot{v}_{G} \in \dot{w}_{G}$, there exist $p \in G$ and $\mu^{\prime} \in M^{\mathbb{P}}$ such that $p \Vdash \varphi\left(\dot{v}, \mu^{\prime}, \dot{u}\right)$ and $\operatorname{ker}\left(\mu^{\prime}\right) \subseteq A$.
Proof. Fix a $\dot{v}_{G} \in \dot{w}_{G}$ for some $\dot{v} \in \operatorname{dom}(\dot{w}) \cap M^{\mathbb{P}}$. There is a $\mathbb{P}$-name $\mu$ and a $p \in G$ such that $p \Vdash \varphi(\dot{v}, \mu, \dot{u}) \wedge$ $\forall z(\varphi(\dot{v}, z, \dot{u}) \rightarrow \mu=z)$.
Claim 38.1. For every $M$-generic filter $H$ over $\mathbb{P}$ such that $p \in H, \operatorname{ker}\left(\mu_{H}\right) \subseteq A$.
Proof of the Claim. Suppose not. Then there is some $b \in \operatorname{ker}\left(\mu_{H}\right) \backslash A$. Since $M$ has a proper class of urelements, there is some urelement $c \in M$ such that $c \notin A \cup \operatorname{ker}(\mu)$. In $M$, let $\pi$ be an automorphism that only swaps $b$ and $c$. Since $\pi$ point-wise fixes $A$, it follows that

$$
p \Vdash \varphi(\dot{v}, \pi \mu, \dot{u}) \wedge \forall z(\varphi(\dot{v}, z, \dot{u}) \rightarrow \pi \mu=z) .
$$

Thus, $M[H] \models \mu_{H}=(\pi \mu)_{H}$. Since $b \in \operatorname{ker}\left(\mu_{H}\right), \pi b \in \operatorname{ker}\left(\pi \mu_{H}\right)$, but $\pi b=c \notin \operatorname{ker}(\mu)$ and $\operatorname{ker}\left(\mu_{H}\right) \subseteq$ $\operatorname{ker}(\mu)$, so $\pi b \notin \operatorname{ker}\left(\mu_{H}\right)$, which is a contradiction.
Note that we cannot hope to show that $\operatorname{ker}(\mu) \subseteq A$ in general. For if $\mu^{*}$ is some $\mathbb{P}$-name such that $\mu^{*}=$ $\mu \cup\left\{\left\langle\left\{\left\langle b, 1_{\mathbb{P}}\right\rangle\right\}, q\right\rangle\right\}$, where $b$ is a urelement not in $A$ and $q$ is incompatible with $p$, we would still have $p \Vdash \mu=\mu^{*}$.

Claim 38.2. Let $H$ be an $M$-generic filter over $\mathbb{P}$ such that $p \in H$. For every $\dot{x}, \dot{y} \in M^{\mathbb{P}}$, if $\dot{x}_{H}, \dot{y}_{H} \in T C\left(\left\{\mu_{H}\right\}\right)$, then $\dot{x}_{H}=\dot{y}_{H}$ if and only if $(\dot{x})_{H}=(\dot{y})_{H}$.

[^1]Proof of the Claim. If $\dot{x}_{H}=\dot{y}_{H}=a$ for some urelement $a$, then by Claim 38.1 $a \in A$. It is easy to check that $\left.\stackrel{A}{(\dot{y}})_{H}=\stackrel{A}{\dot{x}}\right)_{H}=a$. If $\left.(\stackrel{A}{\dot{y}})_{H}=\stackrel{A}{\dot{x}}\right)_{H}=b$ for some urelement $b$, then $b \in A$ and it follows that $\dot{x}_{H}=\dot{y}_{H}=b$.

So suppose $\dot{x}_{H}=\dot{y}_{H}$ are sets in $T C\left(\left\{\mu_{H}\right\}\right)$ and the claim holds for every $\dot{z} \in \operatorname{dom}(\dot{x}) \cup \operatorname{dom}(\dot{y})$. Clearly, $\stackrel{A}{(\dot{x}})_{H}$ and $\left.\stackrel{A}{\dot{y}}\right)_{H}$ must also be sets. If $\stackrel{A}{\dot{z}_{H}} \in \stackrel{A}{\dot{x}_{H}}$ for some $\dot{z} \in M^{\mathbb{P}} \cap \operatorname{dom}(\dot{x})$, we have $\dot{z}_{H} \in \dot{y}_{H}=\dot{x}_{H}$. So there is some $\dot{w} \in M^{\mathbb{P}} \cap \operatorname{dom}(\dot{y})$ such that $\dot{w}_{H}=\dot{z}_{H} . \dot{z}_{H} \in T C\left(\left\{\mu_{H}\right\}\right)$ so by the induction hypothesis $\stackrel{A}{\dot{z}_{H}}=\stackrel{A}{\dot{w}_{H}} \in(\underset{\dot{y}}{\dot{y}})_{H}$. This shows that $\stackrel{A}{\dot{x}_{H}} \subseteq \stackrel{A}{\dot{y}}_{H}$, and we will have $\stackrel{A}{\dot{x}_{H}}=\stackrel{A}{\dot{y}_{H}}$ by the same argument.

Now suppose that $\dot{x}_{H}, \dot{y}_{H} \in T C\left(\left\{\mu_{H}\right\}\right)$ and $\stackrel{A}{\dot{x}_{H}}=\stackrel{A}{\dot{y}_{H}}$ are sets. Then $\dot{x}_{H}$ and $\dot{y}_{H}$ must be sets. For if, say, $\dot{x}_{H}=a$ for some urelement $a$, then $a \in A$ by Claim 38.1. which would yield $\dot{x}_{H}=a$. Let $\dot{z}_{H} \in \dot{x}_{H}$ for some $\dot{z} \in M^{\mathbb{P}} \cap \operatorname{dom}(\dot{x})$. Then $\stackrel{A}{\dot{z}_{H}} \in \stackrel{A}{\dot{y}_{H}}$ and so $\stackrel{A}{\dot{z}_{H}}=\stackrel{A}{\dot{w}_{H}}$ for some $\dot{w}_{H} \in \dot{y}_{H}$. By the induction hypothesis, it follows that $\dot{z}_{H}=\dot{w}_{H}$. This shows that $\dot{x}_{H} \subseteq \dot{y}_{H}$ and consequently, $\dot{x}_{H}=\dot{y}_{H}$.
Claim 38.3. $p \Vdash \stackrel{A}{\mu}=\mu$.
Proof of the Claim. Let $H$ be an $M$-generic filter on $\mathbb{P}$ that contains $p$. We show that $\stackrel{A}{\mu}_{H}=\mu_{H}$. Let $f$ be the function on $T C\left(\left\{\mu_{H}\right\}\right)$ that sends every $\dot{y}_{H}$ to $\dot{\dot{y}}_{H}$, which is is well-defined by Claim 38.2 Note that over $Z F U_{R}$, every $\in$ - isomorphism of transitive sets that fixes the urelements is the identity map. So it suffices to show that $f$ maps $T C\left(\left\{\mu_{H}\right\}\right)$ onto $T C\left(\left\{\mu_{H}^{A}\right\}\right)$, preserves $\in$ and fixes all the urelements.
$f$ preserves $\in$. Consider any $\dot{y}_{H}, \dot{x}_{H} \in T C\left(\left\{\mu_{H}\right\}\right)$. Suppose that $\dot{y}_{H} \in \dot{x}_{H}$. Then $\dot{y}_{H}=\dot{z}_{H}$ for some $\dot{z} \in M^{\mathbb{P}} \cap \operatorname{dom}(\dot{x})$ so $\stackrel{A}{\dot{z}_{H}} \in \stackrel{A}{\dot{x}_{H}}$; by Claim 38.2, it follows that $\stackrel{A}{\dot{y}_{H}}=\stackrel{A}{\dot{z}_{H}} \in \stackrel{A}{\dot{x}_{H}}$. Suppose that $\stackrel{A}{\dot{y}_{H}} \in \stackrel{A}{\dot{x}_{H}}$. Then ${ }_{\dot{y}_{H}}^{A}=\stackrel{A}{\dot{z}}_{H}$ for some $\dot{z}_{H} \in \dot{x}_{H}$ so $\dot{y}_{H}=\dot{z}_{H} \in \dot{x}_{H}$ by Claim 38.2 again.
$f$ maps $T C\left(\left\{\mu_{H}\right\}\right)$ onto $T C\left(\left\{\hat{\mu}_{H}^{A}\right\}\right)$. If $\dot{y}_{H} \in T C\left(\left\{\mu_{H}\right\}\right)$, then $\dot{y}_{H} \in \dot{y}_{1 H} \in \ldots \in \dot{y}_{n H} \in \mu_{H}$ for some $n$. Since $f$ is $\in$-preserving, it follows that $\dot{y}_{H} \in{\stackrel{A}{y_{1}}}_{1_{H}} \in \ldots \in \dot{\bar{y}}_{n_{H}} \in \stackrel{A}{\mu}_{H}$ and hence $\stackrel{A}{\dot{y}}_{H} \in T C\left(\left\{\stackrel{A}{\mu}_{H}\right\}\right)$. To see it is
 and hence $\dot{y}_{H} \in T C\left(\left\{\mu_{H}\right\}\right)$.
$f$ fixes all the urelements in $T C\left(\left\{\mu_{H}\right\}\right)$. Suppose $\dot{x}_{H}=a \in T C\left(\left\{\mu_{H}\right\}\right)$ for some urelement $a$. Then by Claim 38.1, $a \in A$ and hence $\stackrel{A}{\dot{x}_{H}}=a$.
The lemma is now proved by letting $\mu^{\prime}$ be $\stackrel{A}{\mu}$.
Now in M, we define

$$
\bar{x}=\left\{\langle\dot{v}, p\rangle \in\left(\operatorname{dom}(\dot{w}) \cap M^{\mathbb{P}}\right) \times \mathbb{P}: \exists \mu \in M^{\mathbb{P}}(\operatorname{ker}(u) \subseteq A \wedge p \Vdash \varphi(\dot{v}, \mu, \dot{u}))\right\} .
$$

For every $\langle\dot{v}, p\rangle \in \bar{x}$, let $\alpha_{\dot{v}, p}$ be the least $\alpha$ such that there is some $\mu \in V_{\alpha}(A) \cap M^{\mathbb{P}}$ such that $p \Vdash \varphi(\dot{v}, \mu, \dot{u})$. Let $\beta=\operatorname{Sup}_{\langle\dot{v}, p\rangle \in \bar{x}} \alpha_{\dot{v}, p}$ and set $\rho=\left(V_{\beta}(A) \cap M^{\mathbb{P}}\right) \times\left\{1_{\mathbb{P}}\right\}$. It remains to show that $M[G] \mid \forall x \in \dot{w}_{G} \exists y \in$ $\rho_{G} \varphi\left(x, y, \dot{u}_{G}\right)$. Let $\dot{v}_{G} \in \dot{w}_{G}$. By Lemma 38, there is some $p \in G$ such that $\langle\dot{v}, p\rangle \in \bar{x}$. So there is some $\mathbb{P}$-name $\mu \in \operatorname{dom}(\rho)$ such that $p \Vdash \varphi(\dot{v}, \mu, \dot{u})$. Thus, $M[G] \vDash \varphi\left(\dot{v}_{G}, \mu_{G}, \dot{u}_{G}\right)$ and $\mu_{G} \in \rho_{G}$.

For any statement $\varphi$, forcing over countable transitive models of $\mathrm{ZFCU}_{\mathrm{R}}$ preserves $\varphi$ if for every countable transitive model $M \models \mathrm{ZFCU}_{\mathrm{R}}, \varphi$ holds in all generic extensions of $M$ if $\varphi$ holds in $M$.
Theorem 39 (The Fundamental Theorem of Forcing with Urelements). Forcing over countable transitive models of $\mathrm{ZFCU}_{\mathrm{R}}$ preserves the following axioms (and axiom schemes).
(1) All the axioms of $\mathrm{ZFCU}_{\mathrm{R}}$.
(2) Collection.
(3) Plenitude.
(4) Scatter.
(5) Tail.
(6) Closure.
(7) $\mathrm{DC}_{<\text {Ord }}$.

Proof. (1) and (2) are Lemma 34 and Theorem 37. (3) is clear since if Plenitude holds in $M$, then every cardinal $\kappa$ in $M[G]$ is realized by some set of urelements in $M$. (4) follows easily from Lemma 27 ,

Suppose that $M \models \mathrm{ZFCU}_{\mathrm{R}}$ for some $\mathbb{P} \in M$ and fix an $M$-generic filter $G$ over $\mathbb{P}$.
(5) Note first that forcing preserves $\neg$ Plenitude. Suppose for reductio that $M \models \neg$ Plenitude but $M[G] \models$ Plenitude. In $M$, let $\alpha$ be the least cardinal not realized. In $M[G]$, let $\kappa$ be a cardinal above $\alpha$ realized by some set of urelements $A$. By Lemma 27, there is a set of urelements $A^{\prime} \in M$ such that $A \subseteq A^{\prime}$; so in $M$, $A^{\prime}$ is equinumerous with some $\beta<\alpha$. It follows that there is a surjection from $\beta$ onto $\kappa$ in $M[G]$, which is a contradiction. Now suppose $M \models$ Tail. Then $M \models$ Collection by Theorem 7 and since Tail implies $\neg$ Plenitude, it follows that $M[G] \vDash \neg$ Plenitude + Collection, but this implies that Tail holds in $M[G]$ (see Lemma 14 (5)).
(6) Suppose that $M \models$ Closure. Let $X \in M[G]$ be a set of realized cardinals whose supermum is some limit cardinal $\lambda$. Then in $M$, every cardinal $\kappa<\lambda$ is realized. This is because $\lambda$ remains a limit cardinal in $M$ and for every $\kappa<\lambda$, there is some $\kappa^{\prime}$ with $\kappa<\kappa^{\prime}<\lambda$ that is realized in $M[G]$; so it follows from Lemma 27 that $\kappa$ is realized in $M$. By Closure in $M, \lambda$ is realized in $M$ and hence in $M[G]$.
(7) Suppose that $M \models \mathrm{DC}_{<\text {Ord }}$. Since $\mathrm{DC}_{<\text {Ord }}$ implies that either $\mathscr{A}$ is a set, or Plenitude holds, it follows that $M[G] \models(\mathscr{A}$ is a set $\vee$ Plenitude $)$. By Theorem 7 , we have $M[G] \models \mathrm{DC}_{<\text {Ord }}$.

Forcing over countable transitive models of $\mathrm{ZFU}_{\mathrm{R}}$ still preserves all the axioms of $\mathrm{ZFU} \mathrm{U}_{\mathrm{R}}$, Collection, Plenitude, Scatter and $\mathrm{DC}_{<\text {Ord }}\left(\right.$ as $\mathrm{DC}_{<\text {Ord }}$ implies AC). It is unclear, however, if forcing over $\mathrm{ZFU}_{\mathrm{R}}$ preserves Closure and Tail, because the arguments above use the assumption that every set of urelements in the ground model is well-orderable.
4.5. Destroying the $\mathbf{D C}_{\kappa}$-scheme and recovering Collection. I now move on to the preservation of the $\mathrm{DC}_{\kappa}$-scheme. A forcing poset $\mathbb{P}$ is $\kappa$-closed if in $\mathbb{P}$ every infinite descending chain of length less than $\kappa$ has a lowerbound.

Theorem 40. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}+\mathrm{DC}_{\kappa}$-scheme, $\mathbb{P} \in M$ be such that $\left(\mathbb{P} \text { is } \kappa^{+} \text {-closed }\right)^{M}$ and $G$ be an $M$-generic fitler over $\mathbb{P}$. Then $M[G] \models \mathrm{ZFCU}_{\mathrm{R}}+\mathrm{DC}_{\kappa}$-scheme.

Proof. We first make some definitions. For every $\alpha$-sequence $s$ of $\mathbb{P}$-names, let $\dot{s}^{(\alpha)}$ denote the canonical $\mathbb{P}$-name such that $\dot{s}_{G}^{(\alpha)}$ is an $\alpha$-sequence in $M[G]$ with $\dot{s}_{G}^{(\alpha)}(\eta)=s(\eta)_{G}$ for all $\eta<\alpha$. Given a $p \in \mathbb{P}$ and a suitable formula $\varphi$, a $\kappa$-sequence of the form $\left\langle\left\langle p_{\alpha}, \dot{x}_{\alpha}\right\rangle: \alpha<\kappa\right\rangle$, where $\left\langle p_{\alpha}, \dot{x}_{\alpha}\right\rangle \in \mathbb{P} \times M^{\mathbb{P}}$, is said to be a $\varphi$-chain below $p$ if $\left\langle p_{\alpha}: \alpha<\kappa\right\rangle$ is a descending chain below $p$ and for every $\alpha<\kappa, p_{\alpha} \Vdash \varphi\left(\dot{s}^{(\alpha)}, \dot{x}_{\alpha+1}\right)$ where $s=\left\langle\dot{x}_{\eta}: \eta<\alpha\right\rangle$.

Suppose that $M[G] \models \forall x \exists y \varphi(x . y, u)$. There is some $p \in G$ such that $p \Vdash \forall x \exists y \varphi(x, y, \dot{u})$. Let $D$ be the set of forcing conditions that are a lower bound of some $\varphi$-chain below $p$. We claim that $D$ is dense below $p$. If $r \leq p$, let $\psi(x, y, r, \mathbb{P})$ be the formula defined as follows.
$\psi(x, y, \mathbb{P}, \dot{u})={ }_{d f}$ if $x=\left\langle\left\langle p_{\eta}, \dot{x}_{\eta}\right\rangle: \eta<\alpha\right\rangle$, where $\left\langle p_{\eta}: \eta<\alpha\right\rangle$ is a descending chain of length $\alpha$ for some $\alpha<\kappa$, then $y=\langle q, \dot{x}\rangle \in \mathbb{P} \times M^{\mathbb{P}}$ such that $q$ bounds $\left\langle p_{\eta}: \eta<\alpha\right\rangle$ and $q \Vdash \varphi\left(\dot{s}^{(\alpha)}, \dot{x}, \dot{u}\right)$.

Let $\mathbb{P} \downarrow r$ denote the set of conditions in $\mathbb{P}$ below $r$. In $M$, for every $x \in\left(\mathbb{P} \downarrow r \times M^{\mathbb{P}}\right)^{<\kappa}$, since $\mathbb{P}$ is $\kappa$ closed, there is some $y \in \mathbb{P} \downarrow r \times M^{\mathbb{P}}$ such that $\psi(x, y, \mathbb{P} . \dot{u})$. By the $\mathrm{DC}_{\kappa}$-scheme in $M$, there exists a $\varphi$-chain $\left\langle\left\langle p_{\alpha}, \dot{x}_{\alpha}\right\rangle: \alpha<\kappa\right\rangle$, where $\left\langle p_{\alpha}: \alpha<\kappa\right\rangle$ is below $r$ and hence below $p$. $\mathbb{P}$ is $\kappa^{+}$-closed, so there is some $q$ that bounds this $\varphi$-chain below $p$. Thus, $D$ is dense below $p$. It then follows that there is $q \in G$ that bounds a $\varphi$-chain, $\left\langle\left\langle p_{\alpha}, \dot{x}_{\alpha}\right\rangle: \alpha<\kappa\right\rangle$, below $p$. Let $s=\left\langle\dot{x}_{\alpha}: \alpha<\kappa\right\rangle$ and $f=\dot{s}_{G}^{(\kappa)} . f$ is then a $\kappa$-sequence in $M[G]$ and $\kappa$ is not collapsed in $M[G]$ as $\mathbb{P}$ is $\kappa$-closed. Moreover, $M[G] \models \varphi(f \upharpoonright \alpha, f(\alpha), u)$ for all $\alpha<\kappa$ because $q \Vdash \varphi\left(\dot{s}^{(\alpha)}, \dot{x}_{\alpha}, \dot{u}\right)$.

For any infinite cardinals $\kappa$ and $\lambda$ with $\kappa<\lambda, \operatorname{Col}(\kappa, \lambda)$ is the forcing poset consisting of all partial functions from $\kappa$ to $\lambda$ whose domain has size less than $\kappa$ (ordered by reverse inclusion).

Theorem 41. Forcing over countable transitive models of ZFCU does not preserve the $\mathrm{DC}_{\omega_{1}}$-scheme.
Proof. Consider a countable transitive model $M$ of $\mathrm{ZFCU}_{\mathrm{R}}$ where every set of urelements has tail cardinal $\omega_{1}$. By Lemma 11 and Lemma 13, both Collection and the $\mathrm{DC}_{\omega_{1}}$-scheme hold in $M$. Let $\mathbb{P}=\operatorname{Col}\left(\omega, \omega_{1}^{M}\right)$ and $G$ be $M$-generic over $\mathbb{P}$. Then in $M[G]$, every set of urelements is countable, because every $A \in M[G]$ is a subset of some $A^{\prime} \in M$ such that $\left|A^{\prime}\right| \leq \omega_{1}{ }^{M}$ but $\omega_{1}{ }^{M}$ is collapsed to $\omega$ in $M[G]$. As a result, every set of urelements will have tail cardinal $\omega$ in $M[G]$. By an usual argument as in Theorem 18(1), this implies that the $\mathrm{DC}_{\omega_{1}}$-scheme fails in $M[G]$.

Theorem 42. Let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}$ where for every set of urelement, there is another infinite disjoint set of urelements. Then there is a generic extension of $M$ which satisfies ZFCU.
Proof. By Theorem 7 and 34, we may assume that Plenitude fails in $M$ since otherwise Collection holds in every generic extension of $M$. Let $G$ be an $M$-generic filter over $\operatorname{Col}(\omega, \kappa)$, where $\kappa$ is the least cardinal not realized. As $\kappa$ is collapsed to $\omega$ in $M[G]$, every set of urelements in $M[G]$ is countable. If $A$ is a set of urelements in $M[G]$, let $A^{\prime} \in M$ be such that $A \subseteq A^{\prime}$. By the assumption, there is another infinite $B \in M$ disjoint from $A^{\prime}$. Since $B$ has size $\omega$ in $M[G]$, this shows that every set of urelements in $M[G]$ has tail cardinal $\omega$ and so $M[G] \models$ Collection by Lemma 11 .

Corollary 42.1. Every countable transitive model $M$ of $\mathrm{ZFCU}_{\mathrm{R}}+\mathrm{DC}_{\omega}$-scheme has a generic extension that satisfies ZFCU.
Proof. over $\mathrm{ZFCU}_{\mathrm{R}}$, if the $\mathrm{DC}_{\omega}$-scheme holds and there is a proper class of urelements, then for every set of urelements, there is a countably infinite set of urelements disjoint from it. Thus, Theorem 42 applies.

Not every model of $\mathrm{ZFCU}_{\mathrm{R}}$ has a generic extension which satisfies ZFCU. For example, if in $M$ every set of urelements is finite but there is a proper class of them, then this will remain the case in every generic extension of $M$.
4.6. Ground model definability. Laver [12] and Woodin [16] proved independently the ground model definability for ZFC: every transitive model of ZFC is definable in all of its generic extensions with parameters. Laver's argument can be easily adjusted to show that every transitive model of ZFCU with only a set of urelements is definable in all of its generic extensions with parameters ${ }_{3}^{3}$ Here I show that the ground model definability fails badly when the ground model has a proper class of urelements.

For any infinite set of $x, \operatorname{Fn}(x, 2)$ is the forcing poset consisting of all finite partial functions from $x$ to 2 ordered by reversed inclusion. Forcing with $\operatorname{Fn}(x, 2)$ adds a new subset to every set that is equinumerous with $x$.

[^2]Theorem 43. If $M$ is a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}+\mathrm{DC}_{\omega}$-scheme + " $\mathscr{A}$ is a proper class", then $M$ has a generic extension in which $M$ is not definable with parameters.

Proof. Let $\mathbb{P} \in M$ be $\operatorname{Fn}(\omega, 2)$ and $G$ be an $M$-generic filter over $\mathbb{P}$. Suppose for reductio that $M$ is definable in $M[G]$ with a parameter $\dot{u}_{G} \in M[G]$ such that $M=\left\{x \in M[G]: M[G] \models \varphi\left(x, \dot{u}_{G}\right)\right\}$. Let $B^{\prime} \in M$ be an infinite set of urelements disjoint from $\operatorname{ker}(\dot{u})$. Then $M[G]$ contains a new countable subset $B$ of $B^{\prime}$ which is not in $M$. Fix another countably infinite set of urelements $C \in M$ disjoint from $\operatorname{ker}(\dot{u}) \cup B^{\prime}$. In $M[G]$, there will be an automorphism that swaps $C$ and $B$ but point-wise fixes $\operatorname{ker}(\dot{u})$. Since $M[G] \models \neg \varphi\left(B, \dot{u}_{G}\right)$ and $\operatorname{ker}\left(\dot{u}_{G}\right) \subseteq \operatorname{ker}(\dot{u})$, it follows that $M[G] \models \neg \varphi\left(C, \dot{u}_{G}\right)$ and hence $C \notin M$, which is a contradiction.
Theorem 44. If $M$ is a transitive model of ZFCU + Plenitude, then $M$ is not definable in any of its non-trivial generic extensions.
Proof. Consider any $M[G]$ such that $M \subsetneq M[G]$. First observe that there must be some set of urelements $B$ in $M[G] \backslash M$. Fix some $\dot{x}_{G} \in M[G] \backslash M$ of the least rank so that $\dot{x}_{G} \subseteq M$. Let $A=\operatorname{ker}(\dot{x})$. It follows that $\dot{x}_{G} \subseteq V_{\alpha}(A)^{M}$ for some $\alpha$. In $M$, by AC and Plenitude, there is a bijection $f$ from $V_{\alpha}(A)^{M}$ to a set of urelements. $f^{\prime \prime} \dot{x}_{G}$ will then be a set of urelements in $M[G] \backslash M$.

For reductio, suppose that $M=\left\{x \in M[G]: M[G] \models \varphi\left(x, \dot{u}_{G}\right)\right\}$ for some formula $\varphi$ with a parameter $\dot{u}_{G}$. Fix some $B \in M[G] \backslash M$ and $B^{\prime} \in M$ such that $B \subseteq B^{\prime}$. By Plenitude in $M$, there is an $E \in M$ such that $E$ is equinumerous with $B^{\prime}$ and $E$ is disjoint from $\operatorname{ker}(\dot{u})$. So in $M[G]$, there will be a new subset $D \subseteq E$ that is disjoint from $\operatorname{ker}(\dot{u})$. We can again find a $C \in M$ that is equinumerous with $D$ and disjoint from $\operatorname{ker}(\dot{u}) \cup E$. So there will be an automorphism in $M[G]$ that swaps $C$ and $D$ but point-wise fixes $\operatorname{ker}(\dot{u})$. As $M[G] \models \neg \varphi\left(D, \dot{u}_{G}\right)$, it follows that $M[G] \models \neg \varphi\left(C, \dot{u}_{G}\right)$ and hence $C \notin M$, which is a contradiction.

$$
\text { APPENDIX A. } M[G]_{\#}=M[G]
$$

In this appendix, I show that the two generic extensions defined in Definition 22 and 25 are identical. In the following, let $M$ be a countable transitive model of $\mathrm{ZFCU}_{\mathrm{R}}, \mathbb{P} \in M$ be a forcing poset, and $G$ be an $M$-generic filter over $\mathbb{P}$.
Definition 45. For every $\dot{x} \in M_{\#}^{\mathbb{P}}$,

$$
\widetilde{\dot{x}}= \begin{cases}\left\{\left\langle a, 1_{\mathbb{P}}\right\rangle\right\} & \text { if } \mathscr{A}(\dot{x}) \\ \{\langle\widetilde{\dot{y}}, p\rangle:\langle\dot{y}, p\rangle \in \dot{x}\} & \text { otherwise }\end{cases}
$$

Accordingly, the map $\dot{x} \mapsto \tilde{x}$ maps every $\mathbb{P}$-name ${ }_{\#}$ in $M_{\#}^{\mathbb{P}}$ to a $\mathbb{P}$-name in $M^{\mathbb{P}}$. Note that the $G$-valuation is defined differently for names in $M_{\#}^{\mathbb{P}}$ and $M^{\mathbb{P}}$, but this should cause no confusion.
Lemma 46. For every $\dot{y}, \dot{x} \in M_{\#}^{\mathbb{P}}, \dot{y}_{G}=\dot{x}_{G}$ if and only if $\tilde{y}_{G}=\tilde{x}_{G}$.
Proof. We prove it by induction on the rank of $\dot{y}$ and $\dot{x}$. The lemma holds easily when $\dot{y}$ and $\dot{x}$ are urelements. Suppose $\dot{x}$ and $\dot{y}$ are sets. Then $\widetilde{\tilde{x}}$ and $\widetilde{\tilde{y}}$ don't contain any urelements in their domains, so $\widetilde{\dot{y}}_{G}$ and $\widetilde{\dot{x}}_{G}$ must be sets. If $\dot{\sim}_{G} \underset{\sim}{x} \dot{x}_{G}$, then for any $\widetilde{\dot{z}}_{G} \in \widetilde{\dot{y}}_{G}, \dot{z}_{G} \in \dot{\sim}_{G}$ so $\dot{z}_{G}=\dot{v}_{G}$ for some $\dot{v} \in \operatorname{dom}(\dot{\sim})$; by the induction hypothesis, $\widetilde{\dot{z}}_{G}=\widetilde{\dot{v}}_{G} \in \widetilde{\tilde{x}}_{G}$ so $\widetilde{\dot{y}}_{G} \subseteq \widetilde{\dot{x}}_{G}$ and hence $\widetilde{\dot{y}}_{G}=\widetilde{\dot{x}}_{G}$ by the same argument. If $\widetilde{\dot{y}}_{G}=\widetilde{\dot{x}}_{G}$, then for any $\dot{z}_{G} \in \dot{y}_{G}, \widetilde{z}_{G} \in \widetilde{\dot{x}}_{G}$ so $\widetilde{z}_{G}=\widetilde{v}_{G}$ for some $\dot{v} \in \operatorname{dom}(\dot{x})$; by the induction hypothesis, $\dot{z}_{G}=\dot{v}_{G} \in \dot{x}_{G}$; so $\dot{y}_{G} \subseteq \dot{x}_{G}$ and hence the same argument shows that $\dot{y}_{G}=\dot{x}_{G}$.
The next lemma shows that every $\mathbb{P}$-name in $M^{\mathbb{P}}$ is a mixture of the $\sim$-image of some $\mathbb{P}$-names ${ }_{\#}$ in $M_{\#}^{\mathbb{P}}$.
Lemma 47. For every $\dot{x} \in M^{\mathbb{P}}$, there is a function $f: \operatorname{dom}(f) \rightarrow M_{\#}^{\mathbb{P}}$ in $M$ such that
(1) $\operatorname{ker}(f) \subseteq \operatorname{ker}(\dot{x}) \cup \operatorname{ker}(\mathbb{P})$;
(2) $\operatorname{dom}(f)$ is a maximal antichain in $\mathbb{P}$;
(3) for every $p \in \operatorname{dom}(f), p \Vdash \dot{x}=\widetilde{f(p)}$.

Proof. By induction on the rank of $\dot{x}$. Suppose the lemma holds for all the $\mathbb{P}$-names in the domain of $\dot{x}$. Condition (1) allows us to find (without using Collection) some $\alpha$ that is big enough such that for every $\dot{y} \in \operatorname{dom}(\dot{x})$, there is some $f$ as in the lemma that lives in $V_{\alpha}(\operatorname{ker}(\dot{x}) \cup \operatorname{ker}(\mathbb{P}))$. Then by AC in $M$, we can choose a corresponding $f_{\dot{y}}$ for each $\dot{y} \in \operatorname{dom}(\dot{x})$. In $M$, define

$$
\dot{w}=\left\{\left\langle f_{\dot{y}}(p), r\right\rangle: \dot{y} \in \operatorname{dom}(\dot{x}) \cap M^{\mathbb{P}} \wedge \exists q\left(\langle\dot{y}, q\rangle \in \dot{x} \wedge p \in \operatorname{dom}\left(f_{\dot{y}}\right) \wedge r \leq p, q\right)\right\} .
$$

It is clear that $\dot{w} \in M_{\#}^{\mathbb{P}}$ and $\operatorname{ker}(\dot{w}) \subseteq \operatorname{ker}(\dot{x}) \cup \operatorname{ker}(\mathbb{P})$. Define $\mathbb{Q}=\{p \in \mathbb{P}: \exists a \in \mathscr{A} \exists q(\langle a, q\rangle \in \dot{x} \wedge p \leq q)\}$. Let $Y$ be a maximal antichain in $\mathbb{Q}$ and let $X$ be a maximal antichain in $\mathbb{P}$ extending $Y$. Note that for every $p \in Y$, there is a unique urelement $a_{p} \in \operatorname{dom}(\dot{x})$ such that $p \leq q$ and $\left\langle a_{p}, q\right\rangle \in \dot{x}$ for some $q$. Now we define $f: X \rightarrow(\mathscr{A} \cap \operatorname{dom}(\dot{x})) \cup\{\dot{w}\}$ as follows.

$$
f(p)= \begin{cases}a_{p} & \text { if } p \in Y \\ \dot{w} & \text { otherwise }\end{cases}
$$

It is clear that $\operatorname{ker}(f) \subseteq \operatorname{ker}(\dot{x}) \cup \operatorname{ker}(\mathbb{P})$. It remains to show that for every $p \in X, p \Vdash \dot{x}=\widetilde{f(p)}$. Fix a $p \in X$ and an $M$-generic filter $H$ over $\mathbb{P}$ that contains $p$. If $p \in Y$, then $\widetilde{f(p)}=\left\{\left\langle a_{p}, 1_{\mathbb{P}}\right\rangle\right\}$; and since there is a $q$ such that $\left\langle a_{p}, q\right\rangle \in \dot{x}$ and $p \leq q$, it follows that $\dot{x}_{H}=a_{p}=\widetilde{f(p)}{ }_{H}$.
Claim 47.1. If $p \notin Y$, then $\dot{x}_{H}$ is a set.
Proof of the Claim. Suppose $\dot{x}_{H}$ is a urelement. Then for some urelemen $a$ and $q \in H,\langle a, q\rangle \in \dot{x}$. So there is a $r$ which extends both $p$ and $q$; as $r \in \mathbb{Q}$, there is some $s \in Y$ such that $s$ and $r$ are compatible because $Y$ is maximal in $\mathbb{Q}$. But this means that $p$ is compatible with some $s \in Y$, which is a contradiction because $X$ is an antichain.
Suppose that $p \notin Y$. Then $\widetilde{f(p)_{H}}=\widetilde{\dot{w}}_{H}$. Note that $\widetilde{\dot{w}}_{H}$ is a set by the construction of $\widetilde{\dot{w}}$. So by the last claim, it remains to show that $\dot{x}_{H} \subseteq \widetilde{w}_{H}$ and $\widetilde{w}_{H} \subseteq \dot{x}_{H}$. Consider any $\dot{y}_{H} \in \dot{x}_{H}$ with $\langle\dot{y}, q\rangle \in \dot{x}$ and $q \in H$. Since $\operatorname{dom}\left(f_{\dot{y}}\right)$ is a maximal antichain, there is some $p^{\prime} \in \operatorname{dom}\left(f_{\dot{y}}\right)$ and $r \in H$ such that $p^{\prime} \in H$ and $r \leq q, p^{\prime}$. So $\left\langle f_{\dot{y}}\left(p^{\prime}\right), r\right\rangle \in \dot{w}$ and $p^{\prime} \Vdash \dot{y}=\widetilde{f_{\dot{y}}\left(p^{\prime}\right)}$. It follows that $\dot{y}_{H}=\widetilde{f_{\dot{y}}\left(p^{\prime}\right)_{H}} \in \widetilde{\dot{w}}_{H}$ and hence $\dot{x}_{H} \subseteq \widetilde{\dot{w}}_{H}$.

To show that $\widetilde{\dot{w}}_{H} \subseteq \dot{x}_{H}$, fix some $\widetilde{f_{\dot{y}}\left(p^{\prime}\right)_{H}} \in \widetilde{\dot{w}}_{H}$ such that $\dot{y} \in \operatorname{dom}(\dot{x}), p^{\prime} \in \operatorname{dom}\left(f_{\dot{y}}\right)$ and $\left\langle f_{\dot{y}}\left(p^{\prime}\right), r\right\rangle \in \dot{w}$ for some $r \in H$. Then there is some $q$ such that $\langle\dot{y}, q\rangle \in \dot{x}$ and $r \leq p^{\prime}, q$, which implies $\dot{y}_{H} \in \dot{x}_{H}$. As $p^{\prime} \Vdash \dot{y}=\widetilde{f_{\dot{y}}\left(p^{\prime}\right)}$, we have $\widetilde{f_{\dot{y}}\left(p^{\prime}\right)_{H}}=\dot{y}_{H} \in \dot{x}_{H}$, as desired.

Theorem 48. The map $\dot{x}_{G} \mapsto \widetilde{\dot{x}}_{G}$ is an elementary embedding from $M[G]_{\#}$ to $M[G]$. Hence, $M[G]=M[G]_{\#}$.
Proof. We prove it by induction on formulas. Lemma 46 shows that this map is well-defined and 1-1. It is easy to check that the map preserves $\in$ and $\mathscr{A}$. The Boolean cases are trivial. If $M[G] \models \exists x \varphi(x)$, then $M[G] \models \varphi\left(\dot{x}_{G}\right)$ for some $\dot{x} \in M^{\mathbb{P}}$. Fix a function $f$ for $\dot{x}$ as in Lemma 47. Then for some $p \in \operatorname{dom}(f) \cap G$, $p \Vdash \dot{x}=\widetilde{f(p)}$, and so $\dot{x}_{G}=\widetilde{y}_{G}$ where $\dot{y}=f(p) \in M_{\#}^{\mathbb{P}}$. By the induction hypothesis, $M[G]_{\#} \models \varphi\left(\dot{y}_{G}\right)$ and hence $M[G]_{\#} \models \exists x \varphi(x)$. Hence, it follows from Lemma 26 and Theorem 39 that $M[G]_{\#}=M[G]$.

We note that the assumption $M \models \mathrm{AC}$ is not necessary for the conclusion that $M[G]_{\#}=M[G]$. This is because one can show that $M[G]_{\#} \models \mathrm{ZFU}_{\mathrm{R}}$ whenever $M$ does (the argument is the same as the proof of Theorem 37), and so $M[G]_{\#}=M[G]$ by the minimality of both generic extensions. However, the proof used here clarifies the relationship between these two kinds of $\mathbb{P}$-names.

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[^0]:    ${ }^{1}$ The situation here is very similar to the axiomtizations of other fragments of ZFC. For example, in both ZFC without Powerset and intuitionistic ZF, Replacement does not imply Collection over the remaining axioms (see [18] and [3] respectively). And when ZFC without Powerset is formulated with only Replacement, as shown in [5], it turns out to have various pathological models, all of which can be excluded by Collection.

[^1]:    ${ }^{2}$ We view $\pi$ as an automorphism of the background universe.

[^2]:    ${ }^{3}$ In fact, as a corollary of Laver's theorem, one can show that if $M$ is a transitive model of $\mathrm{ZFCU} \mathrm{R}_{\mathrm{R}}$ in which some cardinal $\kappa$ is not realized, $M$ is definable in all of its generic extensions produced by $\kappa$-closed forcing notions. A proof of this can be found in the author's dissertation [17].

