BOOLEAN-VALUED MODELS OF SET THEORY WITH URELEMENTS

XINHE WU, BOKAI YAO

Abstract. We explore Boolean-valued models of set theory with a class of urelements. We begin by examining an existing construction $U^{\mathbb{B}}$ in which every urelement is its own $\mathbb{B}$-name. We prove the fundamental theorem of $U^{\mathbb{B}}$ in the context of $\mathrm{ZFU_R}$ (i.e., ZF with Urelements formulated with Replacement). In particular, we show that $U^{\mathbb{B}}$ preserves Replacement. We also demonstrate how $U^{\mathbb{B}}$ can destroy or recover certain axioms in urelement set theory, such as the Collection Principle. A drawback of $U^{\mathbb{B}}$ is that it does not permit mixing names, resulting in a lack of fullness. To address this, we introduce a new construction, $\overline{U^{\mathbb{B}}}$, which is closed under mixtures. We prove that there is an elementary embedding from $U^{\mathbb{B}}$ to $\overline{U^{\mathbb{B}}}$. Furthermore, over $\mathrm{ZFCU_R}$, $\overline{U^{\mathbb{B}}}$ is full for every complete Boolean algebra $\mathbb{B}$ just in case the Collection Principle holds.

1. Introduction

A Boolean-valued model $M^{\mathbb{B}}$ for a first-order language $\mathcal{L}$ is equipped with a $\mathbb{B}$-valued truth assignment $[\![\ ]\!]$, which assigns a $\mathbb{B}$-value to each sentence in $\mathcal{L}$ in a way that obeys the axioms of first-order logic. In set theory, Boolean-valued models provide an elegant presentation of forcing (see [2], [7], and [6]). In this paper, we study Boolean-valued models of set theory with *urelements*. The earlier studies of forcing with urelements (e.g., [3], [4] and [5]) often assume that all urelements form a set. We explore this topic in a more general setting, where a proper class of urelements is allowed (for poset forcing over transitive models in this setting, see [12]). In Section 2, we begin with a standard construction of Boolean-valued models with urelements, which we call $U^{\mathbb{B}}$. We prove the Fundamental Theorem of $U^{\mathbb{B}}$ (Theorem 2.9) concerning how basic axioms, in particular Replacement, are preserved in each $U^{\mathbb{B}}$ and also show how certain axioms can be destroyed or recovered. In Section 3, we turn to two interesting properties of Boolean-valued models: *mixture* and *fullness*. Since the basic construction of $U^{\mathbb{B}}$ makes it unable to mix different names for urelements, we provide a new construction, $\overline{U^{\mathbb{B}}}$, which is closed under mixtures and proved to be an elementary extension of $U^{\mathbb{B}}$ (Theorem 3.12). Finally, we show that over $\mathrm{ZFCU_R}$, $\overline{U^{\mathbb{B}}}$ is full for every complete Boolean algebra $\mathbb{B}$ just in case the Collection Principle holds (Theorem 3.13). The rest of this section reviews some basic facts and known results about ZF set theory with urelements.

1.1. **Urelement Set Theory.** The first-order language of set theory with urelements $\{\in, \mathscr{A}\}$ contains an additional unary predicate $\mathscr{A}$ for urelements. It is always an axiom that no urelement has any members, and we allow a proper class of urelements. The standard axioms of ZFC will be modified to allow urelements, e.g., Extensionality in this context will assert that *sets* with the same members are identical.

Definition 1.1. ZU is the urelement set theory which includes the following axioms: Extensionality, Foundation, Pairing, Union, Powerset, Infinity, and Separation.
$\mathrm{ZFU_R}$ = ZU + Replacement.
$\mathrm{ZFCU_R}$ = $\mathrm{ZFU_R}$ + Axiom of Choice.
ZF(C) = ZF(C)$\mathrm{U_R}$ + "There is no urelement".

Remark 1.1. The subscript R indicates that the corresponding theories are only formulated with Replacement rather than stronger axioms such as the Collection Principle, i.e.,

$$\text{(Collection) } \forall w, u(\forall x \in w\ \exists y \varphi(x, y, u) \to \exists v \forall x \in w\ \exists y \in v\ \varphi(x, y, u)).$$

Although Collection is provable from $\mathrm{ZFU_R}$ when the urelements form a set, it is not provable from $\mathrm{ZFCU_R}$ when a proper class of urelement is allowed. In particular, $\mathrm{ZFCU_R}$ has *finite-kernel* models, where the urelements form a proper class but every set of them is finite ([13, Theorem 27]). Collection fails in the finite-kernel models because for every $n < \omega$ there is a set of urelements of size n, but we cannot collect them into a set.

1.2. **Notations and facts about $\mathbf{ZFU_R}$.** The symbol "$\mathscr{A}$" will also stand for the class of all urelements; "$A \subseteq \mathscr{A}$" abbreviates "A is a set of urelements"; "U" stands for the class of all objects $\{x \mid x = x\}$. For sets x and y, "$x \sim y$" abbreviates "x is equinumerous with y" and "$x \preceq y$" abbreviates "there is an injection from x into y". For every x, *the kernel of* x, denoted by "$ker(x)$", is the set of urelements in the transitive closure of $\{x\}$. A set is *pure* if its kernel is empty. "V" will denote the class of all pure sets. Ord is the class of all ordinals, which are transitive *pure* sets well-ordered by $\in$. By *cardinals*, we always mean initial ordinals. For any set $A \subseteq \mathscr{A}$, the class $V(A)$ is the cumulative hierarchy built from A by iterating the powerset operation. Namely,

$$V_0(A) = A;$$
$$V_{\alpha+1}(A) = P(V_\alpha(A)) \cup V_\alpha(A);$$
$$V_\gamma(A) = \bigcup_{\alpha<\gamma} V_\alpha(A), \text{ where } \gamma \text{ is a limit ordinal;}$$
$$V(A) = \bigcup_{\alpha\in Ord} V_\alpha(A).$$

For every object x, $ker(x) \subseteq A$ if and only if $x \in V(A)$, and so $U = \bigcup_{A\subseteq\mathscr{A}} V(A)$. Unlike V, when there are urelements U admits definable non-trivial automorphisms. This is because for any definable permutation i of $\mathscr{A}$, i can be extended to an automorphism of U by letting $ix = \{iy \mid y \in x\}$ for every set x; moreover, i point-wise fixes a set x (i.e., $iy = y$ for every $y \in x$) whenever i point-wise fixes $ker(x)$.

In addition to Collection, there is a hierarchy of axioms in urelement set theory studied in [13]. Here we introduce some of them.

Definition 1.2. Let A be a set of urelements. The *tail cardinal of* A, if exists, is the greatest cardinal κ such that there is a set of urelements B of size κ that is disjoint from B.

Intuitively, when the tail cardinal of A exists, A will have a "complement" in the sense of equinumerosity (rather than inclusion).

(Tail) Every set of urelements has a tail cardinal.

(Plenitude) For any cardinal κ, there is a set of urelements of size κ.

(DC_ω-scheme) If for every x there is a y such that $\varphi(x, y, u)$, then there is an ω-sequence $\langle x_n : n < \omega \rangle$ such that $\varphi(x_n, x_{n+1}, u)$ for every n.

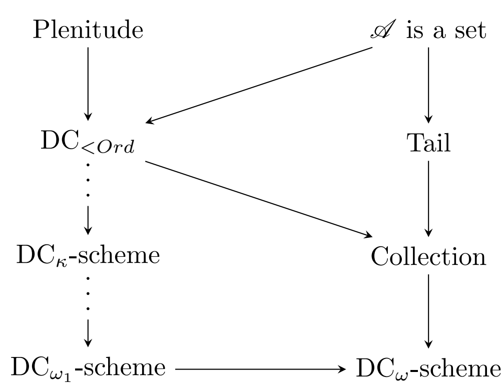
The DC_ω-scheme is a class version of the Axiom of Dependent Choice (DC). Similarly, for any infinite cardinal κ, the DC_κ-scheme generalizes DC_κ as follows.

(DC_κ-scheme) If for every x there is a y such that $\varphi(x, y, u)$, then there is a function f on κ such that $\varphi(f \restriction \alpha, f(\alpha), u)$ for every $\alpha < \kappa$.

$\mathrm{DC}_{<Ord}$ holds if DC_κ-scheme holds for all κ.

Lemma 1.2 ([13, Lemma 22])**.** Let κ be an infinite cardinal. Over $\mathrm{ZFCU_R}$, if every set of urelements has a tail cardinal that is at least κ, then the DC_κ-scheme holds. □

Theorem 1.3 ([13, Theorem 17])**.** Over $\mathrm{ZFCU_R}$, the following implication diagram holds. The diagram is also complete: if the diagram does not indicate φ implies ψ, then $\mathrm{ZFCU_R} + \varphi \nvdash \psi$. □



Note that the finite-kernel models (Remark 1.1) also show that $\mathrm{ZFCU_R}$ cannot prove the DC_ω-scheme, since in these models every set can be properly extended by another set of urelements, but no infinite sequence of increasing sets of urelements exists. Throughout this paper we always work in $\mathrm{ZFU_R}$, unless stated otherwise.

2. $U^{\mathbb{B}}$

2.1. **Basic Facts.** Recall that in ZF, given a complete Boolean algebra $\mathbb{B}$, by transfinite recursion we can define a $\mathbb{B}$-name to be a function from a set of $\mathbb{B}$-names to $\mathbb{B}$. The Boolean-valued universe $V^{\mathbb{B}}$ is then the class of all $\mathbb{B}$-names. In particular, $\emptyset$ will be its own $\mathbb{B}$-name. A natural generalization of this construction in $\mathrm{ZFU_R}$ is to let each urelement be its own $\mathbb{B}$-name, as a different copy of $\emptyset$. This motivates the following definition proposed in [3].

Definition 2.1. Let $\mathbb{B}$ be a complete Boolean algebra.

(1) τ is a $\mathbb{B}$-name if and only if, τ is either a urelement or a function from a set of $\mathbb{B}$-names to $\mathbb{B}$.
(2) $U^{\mathbb{B}} = \{\tau \mid \tau \text{ is a } \mathbb{B}\text{-name}\}$
(3) $\mathcal{L}_{\mathbb{B}}$ is the extended language of urelement set theory containing each $\mathbb{B}$-name as a constant symbol. $\mathcal{AL}_{\mathbb{B}}$ is class of all atomic formulas in $\mathcal{L}_{\mathbb{B}}$.
(4) The Boolean evaluation function $[\![\ \ ]\!] : \mathcal{AL}_{\mathbb{B}} \to \mathbb{B}$ is defined as follows by recursion. For every $\tau, \sigma \in U^{\mathbb{B}}$,

$$[\![\tau \in \sigma]\!] = \bigvee_{\mu \in dom(\sigma)} ([\![\tau = \mu]\!] \wedge \sigma(\mu))$$

$$[\![\tau \subseteq \sigma]\!] = \bigwedge_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![\eta \in \sigma]\!])$$

$$[\![\tau = \sigma]\!] = \begin{cases} 1 & \text{if } \tau, \sigma \in \mathscr{A} \text{ and } \tau = \sigma \\ 0 & \text{if } \tau \in \mathscr{A} \text{ or } \sigma \in \mathscr{A}, \text{ and } \tau \neq \sigma \\ [\![\tau \subseteq \sigma]\!] \wedge [\![\sigma \subseteq \tau]\!] & \text{if } \tau, \sigma \notin \mathscr{A} \end{cases}$$

$$[\![\mathscr{A}(\tau)]\!] = \begin{cases} 1 & \text{if } \tau \in \mathscr{A} \\ 0 & \text{if } \tau \notin \mathscr{A} \end{cases}$$

The evaluation function $[\![\ \ ]\!]$ can be extended into all formulas in $\mathcal{L}_{\mathbb{B}}$ in the standard way. Namely,

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!];$$
$$[\![\neg\varphi]\!] = \neg[\![\varphi]\!];$$
$$[\![\exists x \varphi]\!] = \bigvee_{\tau \in U^{\mathbb{B}}} [\![\varphi(\tau)]\!].$$

We shall also let $U^{\mathbb{B}}$ stand for the Boolean-valued structure $\langle U^{\mathbb{B}}, [\![\ ]\!]\rangle$, where $U^{\mathbb{B}} \models \varphi$ means $[\![\varphi]\!] = 1$. With some trivial modifications of the proofs in [2, p. 24-26], one can show that all the axioms of the first-order logic have value 1 in $U^{\mathbb{B}}$, and all of the rules of inference are valid in $U^{\mathbb{B}}$. As in $V^{\mathbb{B}}$, every object in U has a canonical $\mathbb{B}$-name in $U^{\mathbb{B}}$.

Definition 2.2. For every $x \in U$,

$$\check{x} = \begin{cases} x & x \in \mathscr{A} \\ \{\langle \check{y}, 1\rangle \mid y \in x\} & x \text{ is a set.} \end{cases}$$

It is routine to check that the map $x \mapsto \check{x}$ preserves Δ_0 formulas in the sense that $U \models \varphi(x_1, ..., x_n)$ if and only if $U^{\mathbb{B}} \models \varphi(\check{x_1}, ..., \check{x_n})$ for any Δ_0 formula φ. Consequently, for every Σ_1 formula φ, $U^{\mathbb{B}} \models \varphi(\check{x_1}, ..., \check{x_n})$ whenever $U \models \varphi(x_1, ..., x_n)$. The following facts will be frequently used, and we refer the reader to [2, p.27-47] for their proofs.

Proposition 2.1. For any formula $\varphi(x)$ and any τ in $U^{\mathbb{B}}$,

(1) $[\![\exists x \in \tau\ \varphi(x)]\!] = \bigvee_{\eta \in dom(\tau)} (\tau(\eta) \wedge [\![\varphi(\eta)]\!])$.
(2) $[\![\tau \in Ord]\!] = \bigvee_{\alpha \in Ord} [\![\tau = \check{\alpha}]\!]$.
(3) $[\![\exists x \in Ord\ \varphi(x)]\!] = \bigvee_{\alpha \in Ord} [\![\varphi(\check{\alpha})]\!]$.
(4) (The Induction Principle) $\forall \tau \in U^{\mathbb{B}} (\forall \eta \in dom(\tau) \varphi(\eta) \to \varphi(\tau)) \to \forall \tau \in U^{\mathbb{B}} \varphi(\tau)$. □

The same arguments as in [2, p.37-45] will show that all the axioms of ZU have $\mathbb{B}$-value 1 in $U^{\mathbb{B}}$ and $U^{\mathbb{B}}$ preserves the Axiom of Choice (AC).

Theorem 2.2. $U^{\mathbb{B}} \models$ ZU, and $U^{\mathbb{B}} \models$ AC if $U \models$ AC. □

We conclude this section with a very useful fact, i.e., $U^{\mathbb{B}}$ thinks that every set of urelements is *covered* by some set of urelements in U.

Definition 2.3. For every $\tau \in U^{\mathbb{B}}$, $A_\tau = dom(\tau) \cap \mathscr{A}$.

Lemma 2.3. $[\![\tau \subseteq \mathscr{A}]\!] = [\![\tau \subseteq \check{A}_\tau]\!]$ for every $\tau \in U^{\mathbb{B}}$.

Proof.

$$\begin{aligned}[\![\tau \subseteq \mathscr{A}]\!] &= \bigwedge_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![\mathscr{A}(\eta)]\!]) \\ &= \bigwedge_{\eta \in dom(\tau) \setminus \mathscr{A}} \neg \tau(\eta) \\ &= [\![\tau \subseteq \check{A}_\tau]\!]\end{aligned}$$

□

Corollary 2.3.1. $U^{\mathbb{B}} \models$ "$\mathscr{A}$ is a set" if and only if $U \models$ "$\mathscr{A}$ is a set". □

2.2. **The Fundamental Theorem of $U^{\mathbb{B}}$.** Let $\mathbb{B}$ be a complete Boolean algebra. If φ is some axiom in urelement set theory, it is then natural to ask: does $U^{\mathbb{B}} \models \varphi$ if φ holds in U?

Lemma 2.4. $U^{\mathbb{B}} \models$ Collection if $U \models$ Collection.

Proof. For readability, we shall omit parameters when doing so does not undermine the general idea of the proof. Assume that Collection holds in U. It suffices to show that for every $\tau \in U^{\mathbb{B}}$, there is a $\rho \in U^{\mathbb{B}}$ such that

$$[\![\forall x \in \tau \exists y \varphi(x, y)]\!] \leqslant [\![\forall x \in \tau \exists y \in \rho\ \varphi(x, y)]\!].$$

Now fix $\tau \in U^{\mathbb{B}}$. For any $\sigma \in dom(\tau)$, let $X_\sigma = \{p \in \mathbb{B} \mid \exists \pi \in U^{\mathbb{B}}\ p = [\![\varphi(\sigma, \pi)]\!]\}$. By Collection and Separation in U, it follows that there is a $Y_\sigma \subseteq U^{\mathbb{B}}$ such that $\forall p \in X_\sigma \exists \pi \in Y_\sigma\ p = [\![\varphi(\sigma, \pi)]\!]$. Then $[\![\exists y \varphi(\sigma, x)]\!] = \bigvee_{\pi \in Y_\sigma} [\![\varphi(\sigma, \pi)]\!]$. This shows that for every $\sigma \in dom(\tau)$, there is a $Y_\sigma \subseteq U^{\mathbb{B}}$ such that $[\![\exists y \varphi(\sigma, x)]\!] = \bigvee_{\pi \in Y_\sigma} [\![\varphi(\sigma, \pi)]\!]$. By Collection again, we can collect those Y_σ into a set $\bar{Y}$. Now let ρ be $((\bigcup \bar{Y}) \cap U^{\mathbb{B}}) \times \{1\}$. For any $\sigma \in dom(\tau)$, $[\![\exists y \varphi(\sigma, y)]\!] = \bigvee_{\pi \in \bigcup \bar{Y}} [\![\varphi(\sigma, \pi)]\!] = [\![\exists y \in \rho\ \varphi(\sigma, y)]\!]$. Thus, ρ is as desired. □

The case of Replacement in $U^{\mathbb{B}}$ is more interesting: the standard proof of Replacement having value 1 in $V^{\mathbb{B}}$ uses Collection in V, which does not work for our purpose since Collection is not provable in $\mathrm{ZFCU_R}$. To prove the preservation of Replacement, we shall utilize the idea of *purification*. Intuitively, given a set of urelements A and a $\mathbb{B}$-name τ, the A-purification of τ will be the result of "purifying off" the urelements appeared in the construction of τ that are not in A.

Definition 2.4 (Purification)**.** Let A be a set of urelements and $\tau \in U^{\mathbb{B}}$. We define $\overset{A}{\tau}$, the A-purification of τ, recursively as follows.

(i) $\overset{A}{\tau} = \tau$ if τ is a urelement;

(ii) if τ is a set, then

(a) $dom(\overset{A}{\tau}) = \{\overset{A}{\eta} \mid \eta \in (dom(\tau) \setminus \mathscr{A}) \cup (dom(\tau) \cap A)\}$;

(b) for every $\mu \in dom(\overset{A}{\tau})$,

$$\overset{A}{\tau}(\mu) = \bigvee_{\eta \in X_\mu} \tau(\eta),$$

where $X_\mu = \{\eta \in dom(\tau) \mid \overset{A}{\eta} = \mu\}$.

Example. Let $a \in A$ and $b \notin A$. Suppose that $\mu_1 = \{\langle a, p\rangle, \langle b, q\rangle\}$, $\mu_2 = \{\langle a, p\rangle, \langle b, r\rangle\}$ and $\tau = \{\langle \mu_1, r\rangle, \langle \mu_2, p\rangle, \langle b, p\rangle, \langle a, s\rangle\}$. Then $\overset{A}{\mu_1} = \overset{A}{\mu_2} = \{\langle a, p\rangle\}$ and $\overset{A}{\tau} = \{\left\langle \overset{A}{\mu_1}, r \cup p\right\rangle, \langle a, s\rangle\}$.

Proposition 2.5. Let A be a set of urelements. For every $\tau, \sigma \in U^{\mathbb{B}}$,

(1) $[\![\overset{A}{\tau} \subseteq \tau]\!] = \bigwedge_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!])$.

(2) $[\![\sigma \in \overset{A}{\tau}]\!] = \bigvee_{\eta \in dom(\tau)} (\tau(\eta) \wedge [\![\sigma = \overset{A}{\eta}]\!])$.

Proof. (1) For each $\mu \in dom(\overset{A}{\tau})$, let $X_\mu = \{\eta \in dom(\tau) \mid \overset{A}{\eta} = \mu\}$ and we have

$$\begin{aligned}\overset{A}{\tau}(\mu) \Rightarrow [\![\mu \in \tau]\!] &= (\bigvee_{\eta \in X_\mu} \tau(\eta)) \Rightarrow [\![\mu \in \tau]\!] \\ &= \bigwedge_{\eta \in X_\mu} (\tau(\eta) \Rightarrow [\![\mu \in \tau]\!]) \\ &= \bigwedge_{\eta \in X_\mu} (\tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]).\end{aligned}$$

Thus,

$$\begin{aligned}[\![\overset{A}{\tau} \subseteq \tau]\!] &= \bigwedge_{\mu \in dom(\overset{A}{\tau})} (\overset{A}{\tau}(\mu) \Rightarrow [\![\mu \in \tau]\!]) \\ &= \bigwedge_{\mu \in dom(\overset{A}{\tau})} \bigwedge_{\eta \in X_\mu} (\tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]) \\ &= \bigwedge_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![\overset{A}{\eta} \in \tau]\!]).\end{aligned}$$

(2) For each $\mu \in dom(\overset{A}{\tau})$, $\overset{A}{\tau}(\mu) \wedge [\![\sigma = \mu]\!] = \bigvee_{\eta \in X_\mu} (\tau(\eta) \wedge [\![\sigma = \overset{A}{\eta}]\!])$. So

$$\begin{aligned}[\![\sigma \in \overset{A}{\tau}]\!] &= \bigvee_{\mu \in dom(\overset{A}{\tau})} (\overset{A}{\tau}(\mu) \wedge [\![\sigma = \mu]\!]) \\ &= \bigvee_{\mu \in dom(\overset{A}{\tau})} \bigvee_{\eta \in X_\mu} (\tau(\eta) \wedge [\![\sigma = \overset{A}{\eta}]\!]) \\ &= \bigvee_{\eta \in dom(\tau)} (\tau(\eta) \wedge [\![\sigma = \overset{A}{\eta}]\!]).\end{aligned}$$

□

To prove $U^{\mathbb{B}} \models$ Replacement, a key observation will be that we can compute a lowerbound of $[\![\tau \overset{A}{=} \tau]\!]$ using automorphisms.

Definition 2.5. Let a, b be two urelements. $i^a_b : U \to U$ is the automorphism generated by the permutation of $\mathscr{A}$ which only swaps a and b.

Note that for every set $\tau \in U^{\mathbb{B}}$, $i^a_b\tau = \{\langle i^a_b\eta, i^a_b p\rangle \mid \langle \eta, p\rangle \in \tau\}$, which is a $\mathbb{B}$-name if i^a_b fixes $\mathbb{B}$.

Proposition 2.6. Let a, b be two urelements such that $a, b \notin ker(\mathbb{B})$. For every $\tau, \sigma \in U^{\mathbb{B}}$,

(1) $[\![i^a_b\tau \subseteq \tau]\!] = \bigwedge_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![i^a_b\eta \in \tau]\!])$.

(2) $[\![\sigma \in i^a_b\tau]\!] = \bigvee_{\eta \in dom(\tau)} (\tau(\eta) \wedge [\![\sigma = i^a_b\eta]\!])$.

(3) $[\![\sigma = i^a_b\tau]\!] \leqslant [\![\sigma = \tau]\!]$.

Proof. Note that i^a_b point-wise fixes $\mathbb{B}$. So for every τ and $\eta \in dom(\tau)$, $i^a_b\tau(i^a_b\eta) = \tau(\eta)$, from which (1) and (2) follow.

(3) We prove this by induction. Suppose that the statement holds for all the $\mathbb{B}$-names in $dom(\sigma)$. For every $\gamma \in dom(\sigma)$,

$$
\begin{aligned}
[\![\gamma \in i^a_b\tau]\!] &= \bigvee_{\eta \in dom(\tau)} ([\![\gamma = i^a_b\eta]\!] \wedge \tau(\eta)) \\
\text{(induction hypothesis)} \quad &\leqslant \bigvee_{\eta \in dom(\tau)} ([\![\gamma = \eta]\!] \wedge \tau(\eta)) \\
&= [\![\gamma \in \tau]\!].
\end{aligned}
$$

This shows that

$$\bigwedge_{\gamma \in dom(\sigma)} (\sigma(\gamma) \Rightarrow [\![\gamma \in i^a_b\tau]\!]) \leqslant \bigwedge_{\gamma \in dom(\sigma)} (\sigma(\gamma) \Rightarrow [\![\gamma \in \tau]\!]).$$

Therefore, $[\![\sigma \subseteq i^a_b\tau]\!] \leqslant [\![\sigma \subseteq \tau]\!]$. $[\![i^a_b\tau \subseteq \sigma]\!] \leqslant [\![\tau \subseteq \sigma]\!]$ is shown by the same argument, and hence $[\![\sigma = i^a_b\tau]\!] \leqslant [\![\sigma = \tau]\!]$. □

Lemma 2.7. Let A be a set of urelements such that $ker(\mathbb{B}) \subseteq A$ and a be a urelement not in A. Then for any $\tau \in U^{\mathbb{B}}$,

$$\bigwedge_{b \in \mathscr{A} \setminus A} [\![\tau = i^a_b\tau]\!] \leqslant [\![\tau \overset{A}{=} \tau]\!].$$

Proof. Suppose that the lemma holds for all $\mathbb{B}$-names in $dom(\tau)$.

Claim 2.7.1. For every $\eta \in dom(\tau)$, $[\![\eta \in i^a_b\tau]\!] \wedge [\![i^a_b\eta \in \tau]\!] \leqslant [\![\eta = i^a_b\eta]\!]$.

Proof of the Claim. Let $\eta \in dom(\tau)$. For every $\mu \in dom(\tau)$,

$$
\begin{aligned}
[\![\eta = i^a_b\mu]\!] \wedge [\![\mu = i^a_b\eta]\!] &\leqslant [\![\eta = \mu]\!] \wedge [\![\mu = i^a_b\eta]\!] \\
&\leqslant [\![\eta = i^a_b\eta]\!],
\end{aligned}
$$

where the first line holds by Proposition 2.6 (3). Therefore,

$$
\begin{aligned}
[\![\eta \in i^a_b\tau]\!] \wedge [\![i^a_b\eta \in \tau]\!] &= \bigvee_{\mu \in dom(\tau)} (\tau(\mu) \wedge [\![\eta = i^a_b\mu]\!] \wedge [\![\mu = i^a_b\eta]\!]) \\
&\leqslant [\![\eta = i^a_b\eta]\!],
\end{aligned}
$$

where the first line holds by Proposition 2.6 (2). ■

So we have

$$\begin{aligned}
\bigwedge_{b\in\mathscr{A}\setminus A} [\![\tau = i^a_b\tau]\!] &= \bigwedge_{b\in\mathscr{A}\setminus A} [\![\tau \subseteq i^a_b\tau]\!] \wedge [\![i^a_b\tau \subseteq \tau]\!] \\
\text{(Proposition 2.6 (1))} \quad &= \bigwedge_{b\in\mathscr{A}\setminus A}\ \bigwedge_{\eta\in dom(\tau)} (\tau(\eta) \Rightarrow [\![\eta \in i^a_b\tau]\!] \wedge [\![i^a_b\eta \in \tau]\!]) \\
&= \bigwedge_{\eta\in dom(\tau)} (\tau(\eta) \Rightarrow \bigwedge_{b\in\mathscr{A}\setminus A} [\![\eta \in i^a_b\tau]\!] \wedge [\![i^a_b\eta \in \tau]\!]) \\
\text{(Claim 2.7.1)} \quad &\leqslant \bigwedge_{\eta\in dom(\tau)} (\tau(\eta) \Rightarrow \bigwedge_{b\in\mathscr{A}\setminus A} [\![\eta = i^a_b\eta]\!]) \\
\text{(induction hypothesis)} \quad &\leqslant \bigwedge_{\eta\in dom(\tau)} (\tau(\eta) \Rightarrow [\![\eta = \overset{A}{\eta}]\!]) \\
&= \bigwedge_{\eta\in dom(\tau)} [\tau(\eta) \Rightarrow (\tau(\eta) \wedge [\![\eta = \overset{A}{\eta}]\!] \wedge [\![\overset{A}{\eta} = \eta]\!])] \\
&\leqslant \bigwedge_{\eta\in dom(\tau)} [\tau(\eta) \Rightarrow \bigvee_{\mu\in dom(\tau)} (\tau(\mu) \wedge [\![\eta = \overset{A}{\mu}]\!] \wedge [\![\overset{A}{\eta} = \mu]\!])] \\
\text{(Proposition 2.5 (2))} \quad &= \bigwedge_{\eta\in dom(\tau)} [\tau(\eta) \Rightarrow ([\![\eta \in \overset{A}{\tau}]\!] \wedge [\![\overset{A}{\eta} \in \tau]\!])] \\
\text{(Proposition 2.5 (1))} \quad &= [\![\tau \subseteq \overset{A}{\tau}]\!] \wedge [\![\overset{A}{\tau} \subseteq \tau]\!] \\
&= [\![\tau = \overset{A}{\tau}]\!].
\end{aligned}$$

□

Theorem 2.8. $U^{\mathbb{B}} \models$ Replacement.

Proof. We may assume Collection does not hold in U, otherwise $U^{\mathbb{B}} \models$ Replacement as $U^{\mathbb{B}} \models$ Collection by Lemma 2.4, which means there is a proper class of urelements in U. To prove the theorem, it suffices to show that for every $\pi \in U^{\mathbb{B}}$, there is a $\rho \in U^{\mathbb{B}}$ such that for every $\sigma \in dom(\pi)$,

$$[\![\exists! y\varphi(\sigma, y)]\!] \leqslant [\![\exists y \in \rho\ \varphi(\sigma, y)]\!] \tag{1}$$

Fix a π and let $A = ker(\mathbb{B}) \cup ker(\pi)$.

Claim 2.8.1. For every $\sigma \in dom(\pi)$ and $\tau \in U^{\mathbb{B}}$, there is a $\tau^* \in U^{\mathbb{B}}$ such that $\tau^* \in V(A)$ and $[\![\varphi(\sigma,\tau) \wedge \forall z(\varphi(\sigma, z) \to z = \tau)]\!] \leqslant [\![\varphi(\sigma, \tau^*)]\!]$.

Proof of the Claim. Let $p = [\![\varphi(\sigma,\tau) \wedge \forall z(\varphi(\sigma, z) \to z = \tau)]\!]$ and fix some urelement a such that $a \notin A$, which exists because $\mathscr{A}$ is a proper class. Then for every urelement $b \notin A$, $[\![\varphi(\sigma,\tau)]\!] = [\![\varphi(\sigma, i^a_b\tau)]\!]$, because i^a_b point-wise fixes $\mathbb{B}$ and every $\sigma \in dom(\pi)$. Moreover, for each urelement $b \notin A$,

$$\begin{aligned}
p &\leqslant [\![\varphi(\sigma,\tau)]\!] \wedge ([\![\varphi(\sigma, i^a_b\tau)]\!] \Rightarrow [\![\tau = i^a_b\tau]\!]) \\
&= [\![\varphi(\sigma,\tau)]\!] \wedge [\![\tau = i^a_b\tau]\!] \\
&\leqslant [\![\tau = i^a_b\tau]\!].
\end{aligned}$$

It follows that

$$p \leqslant [\![\varphi(\sigma,\tau)]\!] \wedge \bigwedge_{b\in\mathscr{A}\setminus A} [\![\tau = i^a_b\tau]\!]$$

(Lemma 2.7) $$\leqslant [\![\varphi(\sigma,\tau)]\!] \wedge [\![\tau = \overset{A}{\tau}]\!]$$

$$\leqslant [\![\varphi(\sigma,\overset{A}{\tau})]\!].$$

Since $ker(\overset{A}{\tau}) \subseteq A$, the claim follows. ■

Let $\langle\sigma,p\rangle \in dom(\pi)\times\mathbb{B}$ be such that $p = [\![\varphi(\sigma,\tau)\wedge\forall z(\varphi(\sigma,z)\to z=\tau)]\!]$ for some $\tau\in U^{\mathbb{B}}$. By Claim 2.8.1, there exists a least ordinal $\alpha_{\sigma,p}$ such that $\exists\tau^*\in V_{\alpha_{\sigma,p}}(A)$ with $p \leqslant [\![\varphi(\sigma,\tau^*)]\!]$. Let $\gamma = \bigcup_{\langle\sigma,p\rangle\in dom(\pi)\times\mathbb{B}}\alpha_{\sigma,p}$ and $\rho = (V_\gamma(A)\cap U^{\mathbb{B}})\times\{1\}$. It is easy to check that for every $\sigma\in dom(\pi)$, $[\![\exists! y\varphi(\sigma,y)]\!] \leqslant [\![\exists y\in\rho\ \varphi(\sigma,y)]\!]$, which completes the proof. □

Theorem 2.9 (The Fundamental Theorem of $U^{\mathbb{B}}$)**.** Assume $\mathrm{ZFU_R}$. Let $\mathbb{B}$ be a complete Boolean-algebra. Then

(1) $U^{\mathbb{B}} \models \mathrm{ZFU_R}$, and $U^{\mathbb{B}}\models$ AC if $U\models$ AC;
(2) $U^{\mathbb{B}}\models$ Collection if $U\models$ Collection;
(3) $U^{\mathbb{B}}\models$ Plenitude if $U\models$ Plenitude;
(4) if $U\models$ AC, then $U^{\mathbb{B}}\models$ Plenitude only if $U\models$ Plenitude;
(5) if $U\models$ AC, then $U^{\mathbb{B}}\models$ Tail if $U\models$ Tail.

Proof. (1) is Theorem 2.2 and 2.8. (2) is Lemma 2.4.

(3) Suppose that Plenitude holds in U. It suffices to show that for any $\tau\in U^{\mathbb{B}}$, $[\![\tau\in Ord]\!] \leqslant [\![\exists x\subseteq\mathscr{A}(x\sim\tau)]\!]$. Fix some $\tau\in U^{\mathbb{B}}$. For any ordinal α, there is some $A\subseteq\mathscr{A}$ such that $A\sim\alpha$ and so $[\![\check{A}\sim\check{\alpha}]\!]=1$. Thus, we have

$$[\![\tau=\check{\alpha}]\!] = [\![\tau=\check{\alpha}\wedge\check{\alpha}\sim\check{A}\wedge\check{A}\subseteq\mathscr{A}]\!] \leqslant [\![\check{A}\subseteq\mathscr{A}\wedge\check{A}\sim\tau]\!].$$

This shows that

$$[\![\tau\in Ord]\!] = \bigvee_{\alpha\in Ord}[\![\tau=\check{\alpha}]\!]$$

$$\leqslant \bigvee_{\sigma\in U^{\mathbb{B}}}[\![\sigma\subseteq\mathscr{A}\wedge\sigma\sim\tau]\!]$$

$$= [\![\exists x\subseteq\mathscr{A}(x\sim\tau)]\!].$$

(4) Suppose that $U\models$AC. We prove the stronger statement that $U^{\mathbb{B}}\models\neg$Plenitude if $U\not\models$ Plenitude. Suppose *for reductio* that $U\not\models$ Plenitude but $U^{\mathbb{B}}\not\models\neg$Plenitude. In U, fix some cardinal λ such that there is no set of urelements of size λ. Since $[\![\exists\tau(\tau \text{ is a cardinal } \wedge\check{\lambda}<\tau)]\!]=1$ and $[\![\text{Plenitude}]\!]\neq 0$, it follows that

$$[\![\exists\tau(\tau\text{ is a cardinal }\wedge\check{\lambda}<\tau\wedge\exists\sigma\subseteq\mathscr{A}(\tau\preceq\sigma))]\!]\neq 0.$$

So for some $\tau,\sigma\in U^{\mathbb{B}}$, $[\![\tau\text{ is a cardinal }\wedge\check{\lambda}\in\tau\wedge\tau\preceq\sigma\wedge\sigma\subseteq\mathscr{A}]\!]\neq 0$. By Lemma 2.3, $[\![\sigma\subseteq\mathscr{A}]\!]=[\![\sigma\subseteq\check{A}_\sigma]\!]$, where $A_\sigma=\mathscr{A}\cap dom(\sigma)$. Thus

$$[\![\tau\text{ is a cardinal }\wedge\check{\lambda}\in\tau\wedge\tau\preceq\check{A}_\sigma]\!]\neq 0.$$

By AC in $U^{\mathbb{B}}$, A_σ is equinumerous with some cardinal κ less than λ, so $[\![\check{A}_\sigma\sim\check{\kappa}\wedge\check{\kappa}<\check{\lambda}]\!]=1$. Therefore, $[\![\tau\text{ is a cardinal }\wedge\check{\lambda}<\tau\wedge\check{\kappa}<\check{\lambda}\wedge\tau\preceq\check{\kappa}]\!]\neq 0$, which is a contradiction.

(5) We use the following lemma.

Lemma 2.10. Over $\mathrm{ZFCU_R}$, $\neg$Plenitude $\wedge$ Collection $\rightarrow$ Tail.

Proof. Suppose that Plenitude fails but Collection holds. Given a set $A \subseteq \mathscr{A}$, let κ be the least cardinal such that there is no set of urelements of size κ that is disjoint from A. By Collection, there is a set v such that for every cardinal $\lambda < \kappa$, there is a $B \subseteq \mathscr{A}$ in v such that $B \sim \lambda$ and $B \cap A = \emptyset$. Let $C = \bigcup\{B \in v \mid B \subseteq \mathscr{A}$ and $B \cap A = \emptyset\}$. The cardinality of C is then the tail cardinal of A. □

Now suppose that $U \models \mathrm{AC} \wedge \mathrm{Tail}$. Clearly, $U \not\models$ Plenitude; and $U \models$ Collection by Theorem 1.3. It follows from (2) and the proof of (4) that $U^{\mathbb{B}} \models (\neg\text{Plenitude} \wedge \text{Collection})$. Therefore, $U^{\mathbb{B}} \models$ Tail. □

It is not known if the assumption $U \models \mathrm{AC}$ in Theorem 2.9 (4) and (5) can be dropped. As we will see later (Theorem 2.19), the converse of (5) does not hold: Tail can be recovered in $U^{\mathbb{B}}$ from certain models of $\mathrm{ZFCU_R} + \neg\mathrm{Tail}$.

2.3. **Destroy and Recover Axioms in $U^{\mathbb{B}}$.** We now discuss how $U^{\mathbb{B}}$ can destroy or recover certain axioms (our arguments are analogous to the ones in [12]). The following is a standard construction of the canonical $\mathbb{B}$-name $\dot{f}$ for a sequence f of $\mathbb{B}$-names.

Definition 2.6. For every $\tau, \sigma \in U^{\mathbb{B}}$,

$$\{\tau\}^{\mathbb{B}} = \{\langle \tau, 1\rangle\};$$
$$\{\tau, \sigma\}^{\mathbb{B}} = \{\langle \tau, 1\rangle, \langle \sigma, 1\rangle\};$$
$$\langle \tau, \sigma\rangle^{\mathbb{B}} = \{\{\tau\}^{\mathbb{B}}, \{\tau, \sigma\}^{\mathbb{B}}\}.$$

Let $\alpha \in Ord$ and $f : \alpha \to U^{\mathbb{B}}$ be an α-sequence of $\mathbb{B}$-names.

$$\dot{f} = \{\langle \check{\beta}, f(\beta)\rangle^{\mathbb{B}} \mid \beta \in \alpha\} \times \{1\}.$$

The next proposition is easy to verify based on the definition above.

Proposition 2.11. Let $f : \alpha \to U^{\mathbb{B}}$ be an α-sequence of $\mathbb{B}$-names for some ordinal α and $\beta < \alpha$. Then

(1) $U^{\mathbb{B}} \models \dot{f}$ is an $\check{\alpha}$-sequence;
(2) $U^{\mathbb{B}} \models \langle \check{\beta}, f(\beta)\rangle \in \dot{f}$;
(3) $U^{\mathbb{B}} \models (\dot{f\restriction\beta}) = \dot{f}\restriction\check{\beta}$. □

Let us clarify what the proposition says. According to (1), $U^{\mathbb{B}}$ always thinks that $\dot{f}$ is an $\check{\alpha}$-sequence, and by (2) $U^{\mathbb{B}}$ also thinks that $\dot{f}$ maps $\check{\beta}$ to $f(\beta)$ for every $\beta < \alpha$; (3) shows the uniformity of the canonical names, i.e., for every $\beta < \alpha$, $U^{\mathbb{B}}$ thinks that the canonical name of $f\restriction\beta$, $(\dot{f\restriction\beta})$, is indeed $\dot{f}$ restricted to $\check{\beta}$.

A partial order $(\mathbb{P}, \leqslant)$ is κ-closed if every descending chain in $\mathbb{P}$ of length $\lambda < \kappa$ in $\mathbb{P}$ has a lower bound. A complete Boolean algebra $\mathbb{B}$ is κ-closed if $(\mathbb{B}^+, \leqslant)$ has a dense subset that is κ-closed. It is a classic result in ZFC that every κ-closed $\mathbb{B}$ preserves cardinals below κ, which carries over into $\mathrm{ZFCU_R}$ without any difficulties. In particular, if $\mathbb{B}$ is κ-closed and $\omega_\alpha \leqslant \kappa$, then $U^{\mathbb{B}} \models \check{\omega_\alpha} = \omega_\alpha$. Now we show that κ^+-closed $\mathbb{B}$ will preserve the DC_κ-scheme over $\mathrm{ZFCU_R}$.

Theorem 2.12. Let κ be an infinite cardinal. If $U \models \mathrm{ZFCU_R}+\mathrm{DC}_\kappa$-scheme and $\mathbb{B}$ is κ^+-closed, then $U^{\mathbb{B}} \models \mathrm{DC}_\kappa$-scheme.

Proof. Let $\kappa = \omega_\gamma$ for some ordinal γ. Let $[\![\forall x \exists y \varphi(x,y)]\!] = p$. We show that

$$p \leqslant [\![\exists f (f \text{ is a function on } \omega_\gamma \wedge \forall \alpha < \omega_\gamma\ \varphi(f \restriction \alpha, f(\alpha)))]\!].$$

A κ-sequence of pairs $\langle \langle p_\alpha, \tau_\alpha \rangle : \alpha \in \kappa \rangle$ is said to be *a φ-chain below p* if $\langle p_\alpha : \alpha < \kappa \rangle$ is an infinite descending sequence below p and for every $\alpha < \kappa$, $p_\alpha \leqslant [\![\varphi(\dot{f_\alpha}, \tau_\alpha)]\!]$, where $f_\alpha = \langle \tau_\beta : \beta < \alpha \rangle$. A $q \in \mathbb{B}$ *bounds* $\langle \langle p_\alpha, \tau_\alpha \rangle : \alpha \in \kappa \rangle$ if q is a lower bound of $\langle p_\alpha : \alpha < \kappa \rangle$. Let $X = \{q \in \mathbb{B}^+ : q \text{ bounds some } \varphi\text{-chain below } p\}$.

Claim 2.12.1. X is dense below p.

Proof of the Claim. First, the DC_κ-scheme is equivalent to the following scheme (parameters omitted, see [13, Proposition 13]).

> For every definable class X, if every $s \in X^{<\kappa}$ has some $y \in X$ such that $\varphi(x,y)$, then there is a function $f \in X^\kappa$ such that $\varphi(f \restriction \alpha, f(\alpha))$ for every $\alpha < \kappa$.

Let $0 \neq p' \leqslant p$. Fix some dense subset $D \subseteq \mathbb{B}^+$ that is κ^+-closed and let $D_{\leqslant p'} = \{q \in D \mid q \leqslant p'\}$. Let $\psi(x,y)$ assert

> "If $x = \langle \langle p_\eta, \tau_\eta \rangle : \eta < \alpha \rangle \subseteq (\mathbb{B} \times U^{\mathbb{B}})^\alpha$ for some $\alpha < \kappa$ and $\langle p_\eta : \eta < \alpha \rangle$ is a descending chain, then $y = \langle q, \tau \rangle \in \mathbb{B} \times U^{\mathbb{B}}$ such that q is a lower bound of $\langle p_\eta : \eta < \alpha \rangle$ and $q \leqslant [\![\varphi(\dot{f}, \tau)]\!]$, where $f = \langle \tau_\eta : \eta < \alpha \rangle$."

Now fix some $x = \langle \langle p_\eta, \tau_\eta \rangle : \eta < \alpha \rangle \in (D_{\leqslant p'} \times U^{\mathbb{B}})^{<\kappa}$, where $\langle p_\eta : \eta < \alpha \rangle$ is a descending chain. Let $f = \langle \tau_\eta : \eta < \alpha \rangle$. Since any $\sigma \in U^{\mathbb{B}}$, $p \leqslant \bigvee_{\tau \in U^{\mathbb{B}}} [\![\varphi(\sigma, \tau)]\!]$, $p \leqslant [\![\varphi(\dot{f}, \tau)]\!]$ for some $\tau \in U^{\mathbb{B}}$. $D_{\leqslant p'}$ contains a lower bound q of $\langle p_\eta : \eta < \alpha \rangle$, so there is some $\langle q, \tau \rangle \in D_{\leqslant p'} \times U^{\mathbb{B}}$ such that $\psi(x, \langle q, \tau \rangle)$. By the DC_κ-scheme in U, there exists a φ-chain below p', where all the first components of the pairs are in D. Since D is κ^+-closed, this chain has a bound in X which is $\leqslant p'$. Therefore, X is dense below p. ■

Consider any $q \in X$, which bounds some φ-chain $\langle \langle p_\alpha, \tau_\alpha \rangle : \alpha \in \kappa \rangle$ below p. Let $g = \langle \tau_\alpha : \alpha < \kappa \rangle$. Since $q \leqslant p_\alpha \leqslant [\![\varphi((\dot{g \restriction \alpha}), g(\alpha))]\!]$ for all $\alpha < \kappa$, $q \leqslant \bigwedge_{\alpha < \kappa} [\![\varphi(\dot{g} \restriction \check{\alpha}, \dot{g}(\check{\alpha}))]\!] = [\![\forall \alpha < \check{\kappa}\ \varphi(\dot{g} \restriction \alpha, \dot{g}(\alpha))]\!]$. $\mathbb{B}$ is κ-closed, so $U^{\mathbb{B}} \models \check{\kappa} = \omega_\gamma$. Thus, for every $q \in X$, $q \leqslant [\![\exists f [f \text{ is a function on } \omega_\gamma \wedge \forall \alpha < \omega_\gamma\ \varphi(f \restriction \alpha, f(\alpha))]]\!]$. As X is dense below p, it follows that

$$\begin{aligned} p &\leqslant \bigvee X \\ &\leqslant [\![\exists f [f \text{ is a function on } \omega_\gamma \wedge \forall \alpha < \omega_\gamma \varphi(f \restriction \alpha, f(\alpha))]]\!]. \end{aligned}$$

□

The assumption of Theorem 2.12 cannot be dropped: next we show that there can be some $\mathbb{B}$ such that $U^{\mathbb{B}} \not\models \mathrm{DC}_{\omega_1}$-scheme even if the DC_{ω_1}-scheme holds in U. Given an infinite cardinal κ, the complete Boolean algebra $RO(\kappa^\omega)$ consists of all the regular open sets of the product topology κ^ω, where κ is assigned the discrete topology. It is well-known that in $V^{RO(\kappa^\omega)}$, $\check{\kappa}$ is collapsed to ω. We include a proof of this fact for completeness.

Lemma 2.13. Let κ be an infinite cardinal and $\mathbb{B} = RO(\kappa^\omega)$. Then $U^{\mathbb{B}} \models \check{\kappa} \sim \omega$.

Proof. It suffices to show that $U^{\mathbb{B}} \models \check{\kappa} \leqslant \check{\omega}$. For each $n \in \omega$ and $\xi \in \kappa$, let $p_{n\xi} = \{g \in \kappa^\omega \mid g(n) = \xi\}$. Define $\tau = \{\langle \langle \check{n}, \check{\xi} \rangle^{\mathbb{B}}, p_{n\xi} \rangle \mid n \in \omega \text{ and } \xi \in \kappa\}$. For every $n \in \omega$ and $\xi_1 \neq \xi_2 \in \kappa$, $p_{n\xi_1} \wedge p_{n\xi_2} = \{g \in \kappa^\omega \mid g(n) = \xi_1\} \cap \{g \in \kappa^\omega \mid g(n) = \xi_2\} = \emptyset$. So $U^{\mathbb{B}} \models \tau$ is a partial function on $\check{\omega}$. Also, for any $\xi \in \kappa$,

$\bigvee_{n<\omega} p_{n\xi} = (\overline{\{g \in \kappa^\omega \mid \text{for some } n < \omega, g(n) = \xi\}})^\circ = \kappa^\omega$. So for any $\xi \in \kappa$, $[\![\exists x \in \check{\omega}\tau(x) = \check{\xi}]\!] = \bigvee_{n<\omega} p_{n\xi} = \kappa^\omega$. Therefore, $U^{\mathbb{B}} \models \tau$ is a surjection onto $\check{\kappa}$. □

Lemma 2.14. Suppose that in U every set of urelements has size $\leq \kappa$ for some infinite cardinal κ. Let $\mathbb{B} = RO(\kappa^\omega)$. Then $U^{\mathbb{B}} \models$ every set of urelements is countable.

Proof. For every $\tau \in U^{\mathbb{B}}$,

$$
\begin{aligned}
&\text{(Lemma 2.3)} && [\![\tau \subseteq \mathscr{A}]\!] = [\![\tau \subseteq \check{A}_\tau]\!] \\
&(A_\tau \preceq \kappa) && = [\![\tau \subseteq \check{A}_\tau \wedge \check{A}_\tau \preceq \check{\kappa}]\!] \\
& && \leqslant [\![\tau \preceq \check{\kappa}]\!] \\
&\text{(Lemma 2.13)} && = [\![\tau \preceq \check{\kappa} \wedge \check{\kappa} \sim \omega]\!] \\
& && \leqslant [\![\tau \preceq \omega]\!].
\end{aligned}
$$

That is, $U^{\mathbb{B}} \models$ every set of urelements is countable. □

Lemma 2.15. If every set of urelements is countable and $\mathscr{A}$ is a proper class, then the DC_{ω_1}-scheme fails.

Proof. For every x there is a y with $ker(x) \subsetneq ker(y)$, but there cannot be a function f on ω_1 with $ker(f \restriction \alpha) \subsetneq ker(f(\alpha))$ for all $\alpha < \omega_1$ otherwise f would have an uncountable kernel. □

Theorem 2.16. There is a model U of $\mathrm{ZFCU_R}$ and some $\mathbb{B} \in U$ such that $U^{\mathbb{B}} \not\models$ DC_{ω_1}-scheme but $U \models \mathrm{DC}_{\omega_1}$-scheme $\wedge$ Collection.

Proof. Consider a model U of $\mathrm{ZFCU_R}$ where every set of urelements has tail cardinal ω_1 (see [13, Theorem 27 (6)]) and let $\mathbb{B} = RO(\omega_1^\omega)$. By Theorem 1.3 and 1.2, both the DC_{ω_1}-scheme and Collection hold in U. By Lemma 2.14, $U^{\mathbb{B}} \models$ every set of urelements is countable. Therefore, $U^{\mathbb{B}} \not\models \mathrm{DC}_{\omega_1}$-scheme by Lemma 2.15. □

When $U \models \mathrm{ZFCU_R}$ + Collection + DC_ω-scheme, $U^{\mathbb{B}} \models \mathrm{DC}_\omega$-scheme for every $\mathbb{B}$ because the DC_ω-scheme follows from $\mathrm{ZFCU_R}$ + Collection (Theorem 1.3). The following question, however, is open.

Question 2.17. If $U \models \mathrm{ZFCU_R} + \mathrm{DC}_\omega$-scheme, does $U^{\mathbb{B}} \models \mathrm{DC}_\omega$-scheme for every $\mathbb{B}$?

$RO(\kappa^\omega)$ can be used to recover Collection and Tail from certain models of $\mathrm{ZFCU_R}$.

Theorem 2.18. Suppose that $U \models \mathrm{ZFCU_R}$ and in U for every set of urelements there is an infinite set of urelements disjoint from it. Then for some $\mathbb{B} \in U$, $U^{\mathbb{B}} \models$ Collection.

Proof. We may assume that Plenitude fails in U otherwise by Theorem 1.3 and Theorem 2.9, every $U^{\mathbb{B}}$ satisfies Collection. Thus, there is a least cardinal κ in U such that there is no set of urelements of size κ. Let $\mathbb{B} = RO(\kappa^\omega)$. By Lemma 2.14, $U^{\mathbb{B}} \models$ every set of urelements is countable. Moreover, for every $\tau \in U^{\mathbb{B}}$, let $B \in U$ be an infinite set of urelements disjoint from A_τ; since $[\![\check{A}_\tau \cap \check{B} = \emptyset]\!] = 1$, $[\![\tau \subseteq \mathscr{A}]\!] = [\![\tau \subseteq \check{A}_\tau]\!] \leqslant [\![\tau \cap \check{B} = \emptyset \wedge \check{B} \text{ is infinite}]\!]$. Therefore, $U^{\mathbb{B}} \models$ Tail because $U^{\mathbb{B}}$ thinks that every set of urelements has tail cardinal ω. It then follows from Theorem 1.3 that $U^{\mathbb{B}} \models$ Collection. □

The assumption of this theorem cannot be dropped: if U is a finite-kernel model (Remark 1.1) then every $U^{\mathbb{B}}$ will be a finite-kernel model by Lemma 2.3 and hence $U^{\mathbb{B}} \models \neg$Collection.

Corollary 2.18.1. If $U \models \mathrm{ZFCU_R}+ \mathrm{DC}_\omega$-scheme, then $U^{\mathbb{B}} \models$ Collection for some $\mathbb{B} \in U$.

Proof. We may assume $\mathscr{A}$ is a proper class in U. Then by DC_ω-scheme, for every set A of urelements, there is an infinite sequence of sets of urelements $\langle A_n : n < \omega\rangle$ such that $A_n \subsetneq A_{n+1}$ and $A_n \cap A = \emptyset$. The kernel of this sequence will be an infinite set of urelements disjoint from A, so Theorem 2.18 applies. □

Theorem 2.19. There is a model U of $\mathrm{ZFCU_R}$ + ¬Tail such that $U^{\mathbb{B}} \models$ Tail for some $\mathbb{B} \in U$.

Proof. Let U be a model of $\mathrm{ZFCU_R}$ where every set of urelements has size $< \aleph_\omega$ and for every $n < \omega$, there is some $A \subseteq \mathscr{A}$ of size $\aleph_n$. The existence of U follows from [13, Theorem 26]. Thus, $U \models \neg$Tail. Let $\mathbb{B} = RO((\aleph_\omega)^\omega)$. By Lemma 2.14 and the proof of Theorem 2.18, $U^{\mathbb{B}} \models$ Tail. □

3. A New Construction $\overline{U^{\mathbb{B}}}$

3.1. $U^{\mathbb{B}}$ Is Almost Never Full.

A very desirable property of Boolean-valued models is called *fullness*.

Definition 3.1. Let $M^{\mathbb{B}}$ be a Boolean-valued model. $M^{\mathbb{B}}$ is *full* if and only if, for any formula $\varphi(v, v_1, ..., v_n)$ and $\tau_1, ..., \tau_n \in M^{\mathbb{B}}$, there is some $\tau \in M^{\mathbb{B}}$ such that $[\![\exists v\varphi(v, \tau_1, ..., \tau_n)]\!] = [\![\varphi(\tau, \tau_1, ..., \tau_n)]\!]$.

In other words, a Boolean-valued model is full just in case there exists a "witness" in the model for every existential formula. It is a standard result that if $V \models$ ZFC, then $V^{\mathbb{B}}$ is full for every $\mathbb{B}$. One reason why fullness is highly desirable is because the generalized Łoś theorem holds for all full Boolean-valued models (see [9], [8], [1], and [11] for many other applications of fullness). In the context of set theory, this allows us to construct forcing extensions over any model of ZFC (see [6] for more on this).

However, the very construction of $U^{\mathbb{B}}$ makes it almost never full.

Remark 3.1. $U^{\mathbb{B}}$ is not full if there are two urelements and $\mathbb{B}$ is a proper extension of $\mathbb{2}$.

Proof. Consider the $\mathbb{B}$-name $\tau = \{\langle a_1, p\rangle, \langle a_2, \neg p\rangle\}$, where a_1, a_2 are two different urelements and p is an intermediate Boolean value. Let $\varphi(v)$ be the formula $\exists x(\mathscr{A}(x) \wedge x \in v)$. Then $[\![\varphi(\tau)]\!] = 1$. Suppose *for reductio* that there is some $\sigma \in U^{\mathbb{B}}$ with $[\![\mathscr{A}(\sigma) \wedge \sigma \in \tau]\!] = 1$. By Definition 2.1, σ must be identical to both a_1 and a_2—contradiction. □

One root of this drawback is that $U^{\mathbb{B}}$ is not closed under *mixtures* of names, the general notion of which is defined as follows.

Definition 3.2. Let $M^{\mathbb{B}}$ be a Boolean-valued model, $\{\sigma_i \mid i \in I\} \subseteq M^{\mathbb{B}}$ and $\{p_i \mid i \in I\} \subseteq \mathbb{B}$, where I is an index set. $\tau \in M^{\mathbb{B}}$ *mixes* $\{\sigma_i \mid i \in I\}$ with respect to $\{p_i \mid i \in I\}$ if $p_i \leqslant [\![\tau = \sigma_i]\!]^{M^{\mathbb{B}}}$ for every $i \in I$. τ is also said to be their *mixture*.

It is a standard result in ZF that $V^{\mathbb{B}}$ is closed under mixtures.

Lemma 3.2. Assume ZF in V and let $\mathbb{B} \in V$ be a complete Boolean algebra. Every $\{\tau_i \mid i \in I\} \subseteq V^{\mathbb{B}}$ and antichain $\{p_i \mid i \in I\} \subseteq \mathbb{B}$ have a mixture.

Proof. This is by a standard argument as in [2, Mixing Lemma 1.25]. Let $dom(\tau) = \bigcup_{i\in I} dom(\tau_i)$. For every $\sigma \in dom(\tau)$, let $\tau(\sigma) = \bigvee_{i\in I}(p_i \wedge [\![\sigma \in \tau_i]\!])$. Note that the definition makes sense since all names in $V^{\mathbb{B}}$ are sets. It is routine to check τ is indeed a mixture. $\square$

$U^{\mathbb{B}}$, as we have seen, is not closed under mixtures since no name can mix two different urelements indexed by two incompatible intermediate values.

3.2. **A New Construction.** It is natural to ask if there can be a different construction of Boolean-valued models with urelements that is closed under mixtures and if these models will be full. In this subsection we provide a new construction of names that is closed under mixtures. The issue of fullness will be addressed in the last subsection.

Definition 3.3. Let $\mathbb{B}$ be a complete Boolean algebra.

(1) $\tau : dom(\tau) \to \mathbb{B}$ is a $\mathbb{B}$-*mixed-name* if and only if, for any $x \in dom(\tau)$, x is either a urelement or a $\mathbb{B}$-mixed-name, and for any urelement $a \in dom(\tau)$ and $x \in dom(\tau)$ such that $x \neq a$, $\tau(a) \wedge \tau(x) = 0$.
(2) $\overline{U^{\mathbb{B}}} = \{\tau : \tau \text{ is a } \mathbb{B}\text{-mixed-name}\}$.
(3) Let $\tau \in \overline{U^{\mathbb{B}}}$. $dom^{\mathscr{A}}(\tau) = dom(\tau) \cap \mathscr{A}$; $dom^{\mathbb{B}}(\tau) = dom(\tau) \cap \overline{U^{\mathbb{B}}}$.
(4) $\mathcal{L}_{\overline{\mathbb{B}}}$ is the extended language that contains an additional binary predicate $\overset{\mathscr{A}}{=}$ and each $\mathbb{B}$-mixed-name as a constant symbol. $\mathcal{AL}_{\overline{\mathbb{B}}}$ is the class of all atomic formulas in $\mathcal{L}_{\overline{\mathbb{B}}}$. The Boolean evaluation function $[\![\ ]\!]^{\overline{U^{\mathbb{B}}}} : \mathcal{AL}_{\overline{\mathbb{B}}} \to \mathbb{B}$ is defined as follows.

$$[\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} = \bigvee_{a\in dom^{\mathscr{A}}(\tau)} \tau(a);$$

$$[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a\in dom^{\mathscr{A}}(\tau)\cap dom^{\mathscr{A}}(\sigma)} (\tau(a) \Leftrightarrow \sigma(a)) \ \wedge \bigwedge_{a\in dom^{\mathscr{A}}(\tau)\setminus dom^{\mathscr{A}}(\sigma)} \neg\tau(a)$$
$$\wedge \bigwedge_{a\in dom^{\mathscr{A}}(\sigma)\setminus dom^{\mathscr{A}}(\tau)} \neg\sigma(a);$$

$$[\![\tau \in \sigma]\!]^{\overline{U^{\mathbb{B}}}} = \bigvee_{\mu\in dom^{\mathbb{B}}(\sigma)} ([\![\tau = \mu]\!]^{\overline{U^{\mathbb{B}}}} \wedge \sigma(\mu));$$

$$[\![\tau \subseteq \sigma]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{\eta\in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \Rightarrow [\![\eta \in \sigma]\!]^{\overline{U^{\mathbb{B}}}});$$

$$[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} = [\![\tau \subseteq \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}}.$$

$[\![\ ]\!]^{\overline{U^{\mathbb{B}}}}$ can be extended to all formulas in $\mathcal{L}_{\overline{\mathbb{B}}}$ in the standard way as before, and we shall let $\overline{U^{\mathbb{B}}}$ denote the structure $\left\langle \overline{U^{\mathbb{B}}}, [\![\ ]\!]^{\overline{U^{\mathbb{B}}}} \right\rangle$.

Let us explain the idea behind mixed-names and highlight some differences between $U^{\mathbb{B}}$ and $\overline{U^{\mathbb{B}}}$. First, no urelement is a $\mathbb{B}$-mixed-name since every $\mathbb{B}$-mixed-name is a set, and each urelement a will be represented by $\{\langle a, 1\rangle\}$ in $\overline{U^{\mathbb{B}}}$ instead of itself. Second, for any $\tau \in \overline{U^{\mathbb{B}}}$ and urelement a, $\tau(a)$, metaphorically speaking, represents the $\mathbb{B}$-value of τ "being identical to" the urelement a, rather than the value of a

"being a member of" τ. Accordingly, there can be mixed-names that is identical to both (the canonical names of) a set and a urelement to a non-zero degree, e.g., $\{\langle a, p\rangle, \langle\emptyset, \neg p\rangle\}$. And $[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}}$ is the degree to which τ and σ are identical given that they are taken as urelements.

Finally, the incompatibility condition in Definition 3.3 (1) amounts to the restriction that for every $\sigma \in dom^{\mathbb{B}}(\tau)$, $\{\tau(a) \mid a \in dom^{\mathscr{A}}(\tau)\} \cup \{\tau(\sigma)\}$ is an antichain, which is motivated by the following consideration. If $a, b \in dom(\tau)$ are two urelements, then $\tau(a) \wedge \tau(b)$ must be 0 because this is the degree to which τ is both of them; if $a, \sigma \in dom(\tau)$, where σ is a $\mathbb{B}$-mixed-name, then $\tau(a) \wedge \tau(\sigma)$ must be 0 because this is the degree to which τ is a urelement with a member. In fact, this restriction ensures that no urelement can have any members in $\overline{U^{\mathbb{B}}}$.

Proposition 3.3. $[\![\forall x(\mathscr{A}(x) \to \forall y(y \notin x))]\!]^{\overline{U^{\mathbb{B}}}} = 1.$

Proof. For every $\tau \in \overline{U^{\mathbb{B}}}$,

$$\begin{aligned} [\![\mathscr{A}(\tau)]\!] &= \bigvee_{a \in dom^{\mathscr{A}}(\tau)} \tau(a) \\ &\leqslant \bigwedge_{\mu \in dom^{\mathbb{B}}(\tau)} \neg\tau(\mu) \\ &\leqslant \bigwedge_{\sigma \in U^{\mathbb{B}}} \bigwedge_{\mu \in dom^{\mathbb{B}}(\tau)} ([\![\sigma \neq \eta]\!] \vee \neg\tau(\mu)) \\ &= [\![\forall y(y \notin \tau)]\!]. \end{aligned}$$

The second line holds because for any urelement $a \in dom^{\mathscr{A}}(\tau)$ and $\mu \in dom^{\mathbb{B}}(\tau)$, $\tau(a) \leq \neg\tau(\mu)$. ☐

Next we verify that $\overline{U^{\mathbb{B}}}$ is indeed a Boolean-valued model.

Proposition 3.4. For any τ, σ, π in $\overline{U^{\mathbb{B}}}$,

(1) $[\![\tau = \tau]\!]^{\overline{U^{\mathbb{B}}}} = 1$;
(2) $\tau(\eta) \leqslant [\![\eta \in \tau]\!]^{\overline{U^{\mathbb{B}}}}$ for every $\eta \in dom^{\mathbb{B}}(\tau)$;
(3) $[\![\tau = \sigma]\!]^{\overline{U^{\mathbb{B}}}} = [\![\sigma = \tau]\!]^{\overline{U^{\mathbb{B}}}}$;
(4) $[\![\tau = \sigma \wedge \tau \in \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\sigma \in \pi]\!]^{\overline{U^{\mathbb{B}}}}$;
(5) $[\![\tau = \sigma \wedge \pi \in \tau]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\pi \in \sigma]\!]^{\overline{U^{\mathbb{B}}}}$;
(6) $[\![\tau = \sigma \wedge \sigma = \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\tau = \pi]\!]^{\overline{U^{\mathbb{B}}}}$;
(7) $[\![\tau = \sigma \wedge \mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\mathscr{A}(\sigma)]\!]^{\overline{U^{\mathbb{B}}}}$.

Proof. The proofs of (1)-(5) can be found in [2, Theorem 1.23] with trivial modifications, as $[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}}$ causes no difficulties for the usual proofs to go through.

(6) The proof of $[\![\tau = \sigma \wedge \sigma = \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\tau \subseteq \pi]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\pi \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}}$ is exactly the same as in [2, p.31], so we only show $[\![\tau = \sigma \wedge \sigma = \pi]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\tau \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}}$. Let

$$\begin{aligned} \tau^+ &= \tau \cup \{\langle a, 0\rangle \mid a \in dom^{\mathscr{A}}(\sigma) \cup dom^{\mathscr{A}}(\pi) \text{ and } a \notin dom^{\mathscr{A}}(\tau)\}; \\ \sigma^+ &= \sigma \cup \{\langle a, 0\rangle \mid a \in dom^{\mathscr{A}}(\tau) \cup dom^{\mathscr{A}}(\pi) \text{ and } a \notin dom^{\mathscr{A}}(\sigma)\}; \\ \pi^+ &= \pi \cup \{\langle a, 0\rangle \mid a \in dom^{\mathscr{A}}(\tau) \cup dom^{\mathscr{A}}(\sigma) \text{ and } a \notin dom^{\mathscr{A}}(\pi)\}. \end{aligned}$$

τ^+, σ^+ and π^+ then have the same set of urelements in their domain, so let $A = dom^{\mathscr{A}}(\tau^+)$. It is easy to see that

$[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in A}(\tau^+(a) \Leftrightarrow \sigma^+(a))$;
$[\![\sigma \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in A}(\sigma^+(a) \Leftrightarrow \pi^+(a))$; and
$[\![\tau \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in A}(\tau^+(a) \Leftrightarrow \pi^+(a))$.

Thus,

$$\begin{aligned} [\![\tau = \sigma \wedge \sigma = \pi]\!]^{\overline{U^{\mathbb{B}}}} &\leqslant [\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\sigma \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}} \\ &= \bigwedge_{a \in A}(\tau^+(a) \Leftrightarrow \sigma^+(a)) \wedge \bigwedge_{a \in A}(\sigma^+(a) \Leftrightarrow \pi^+(a)) \\ &\leqslant \bigwedge_{a \in A}(\tau^+(a) \Leftrightarrow \pi^+(a)) \\ &= [\![\tau \overset{\mathscr{A}}{=} \pi]\!]^{\overline{U^{\mathbb{B}}}}. \end{aligned}$$

(7) We only show $[\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\mathscr{A}(\sigma)]\!]^{\overline{U^{\mathbb{B}}}}$. Let τ^+, σ^+ and A be as in the previous paragraph. It follows that

$$\begin{aligned} [\![\tau \overset{\mathscr{A}}{=} \sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} &= \bigwedge_{a \in A}(\tau^+(a) \Leftrightarrow \sigma^+(a)) \wedge \bigvee_{a \in A} \tau^+(a) \\ &\leqslant \bigvee_{a \in A} \sigma^+(a) = \bigvee_{a \in dom^{\mathscr{A}}(\sigma)} \sigma(a) = [\![\mathscr{A}(\sigma)]\!]^{\overline{U^{\mathbb{B}}}}. \end{aligned}$$

□

Proposition 3.5. For every τ in $\overline{U^{\mathbb{B}}}$,

$$[\![\exists x \in \tau \ \varphi(x)]\!]^{\overline{U^{\mathbb{B}}}} = \bigvee_{\eta \in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \wedge [\![\varphi(\eta)]\!]^{\overline{U^{\mathbb{B}}}}).$$

Proof. The proof is exactly the same as in [2, p.27]. One just need to note that the urelements in $dom(\tau)$, if any, will be ignored when computing $[\![\sigma \in \tau]\!]^{\overline{U^{\mathbb{B}}}}$. □

We now show that $\overline{U^{\mathbb{B}}}$ is closed under mixtures.

Lemma 3.6. Every $\{\tau_i \mid i \in I\} \subseteq \overline{U^{\mathbb{B}}}$ and antichain $\{p_i \mid i \in I\} \subseteq \mathbb{B}$ have a mixture.

Proof. We define their mixture τ as follows. Let $dom(\tau) = \bigcup_{i \in I} dom(\tau_i)$. For every $x \in dom(\tau)$,

$$\tau(x) = \bigvee_{i \in J_x} (p_i \wedge \tau_i(x)),$$

where $J_x = \{i \in I \mid x \in dom(\tau_i)\}$.

Claim 3.6.1. $\tau \in \overline{U^{\mathbb{B}}}$.

Proof of the Claim. Suppose that $a \in dom^{\mathscr{A}}(\tau)$ and $x \in dom(\tau)$ with $x \neq a$. Then $\tau(a) \wedge \tau(x) = \bigvee_{i \in J_a}(p_i \wedge \tau_i(a)) \wedge \bigvee_{j \in J_x}(p_j \wedge \tau_j(x))$. We need to show that for any $i \in J_a, j \in J_x$,

$$p_i \wedge \tau_i(a) \wedge p_j \wedge \tau_j(x) = 0.$$

If $i \neq j$, then $p_i \wedge p_j = 0$ as $\{p_i \mid i \in I\}$ is an antichain. If $i = j$, then $a, x \in dom(\tau_i)$ so $\tau_i(a) \wedge \tau_i(x) = 0$ because τ_i is a $\mathbb{B}$-mixed-name. ■

Now we show that for every $i \in I$, $p_i \leqslant [\![\tau = \tau_i]\!]^{\overline{U^{\mathbb{B}}}}$. Fix an $i \in I$.

Claim 3.6.2. $p_i \leqslant \llbracket \tau \subseteq \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}}$.

Proof of the Claim. Let $\eta \in dom^{\mathbb{B}}(\tau)$ and $j \in J_\eta$. If $i \neq j$, then $p_i \leqslant \neg p_j$; otherwise, $\eta \in dom(\tau_i)$ so $\tau_i(\eta) \leqslant \llbracket \eta \in \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}}$ by Proposition 3.4 (2), and hence $\neg\tau_j(\eta) \vee \llbracket \eta \in \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}} = 1$. It follows that for every $j \in J_\eta$, $p_i \leqslant \neg p_j \vee \neg\tau_j(\eta) \vee \llbracket \eta \in \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}}$. Thus, for every $\eta \in dom^{\mathbb{B}}(\tau)$,

$$\begin{aligned} p_i &\leqslant [\bigvee_{j \in J_\eta} p_i \wedge \tau_j(\eta)] \Rightarrow \llbracket \eta \in \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}} \\ &= \tau(\eta) \Rightarrow \llbracket \eta \in \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}}. \end{aligned}$$

Therefore, $p_i \leqslant \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \Rightarrow \llbracket \eta \in \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}})$ so $p_i \leqslant \llbracket \tau \subseteq \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}}$. ■

Claim 3.6.3. $p_i \leqslant \llbracket \tau_i \subseteq \tau \rrbracket^{\overline{U^{\mathbb{B}}}}$.

Proof of the Claim. For every $\eta \in dom^{\mathbb{B}}(\tau_i)$,

$$\begin{aligned} p_i &\leqslant \tau_i(\eta) \Rightarrow (p_i \wedge \tau_i(\eta)) \\ &\leqslant \tau_i(\eta) \Rightarrow \tau(\eta) \\ &\leqslant \tau_i(\eta) \Rightarrow \llbracket \eta \in \tau \rrbracket^{\overline{U^{\mathbb{B}}}}. \end{aligned}$$

Therefore, $p_i \leqslant \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau_i)} (\tau_i(\eta) \Rightarrow \llbracket \eta \in \tau \rrbracket^{\overline{U^{\mathbb{B}}}})$ so $p_i \leqslant \llbracket \tau_i \subseteq \tau \rrbracket^{\overline{U^{\mathbb{B}}}}$. ■

Claim 3.6.4. $p_i \leqslant \llbracket \tau \overset{\mathscr{A}}{=} \tau_i \rrbracket^{\overline{U^{\mathbb{B}}}}$.

Proof of the Claim. By the definition of $\overset{\mathscr{A}}{=}$ and τ, it suffices to show the following two hold.

$$\forall a \in dom^{\mathscr{A}}(\tau) \setminus dom^{\mathscr{A}}(\tau_i),\ p_i \leqslant \neg\tau(a). \tag{2}$$

$$\forall a \in dom^{\mathscr{A}}(\tau) \cap dom^{\mathscr{A}}(\tau_i),\ p_i \leqslant \tau(a) \Leftrightarrow \tau_i(a). \tag{3}$$

For (2), for every $a \in dom^{\mathscr{A}}(\tau) \setminus dom^{\mathscr{A}}(\tau_i)$, $i \notin J_a$ and so $p_i \leqslant \neg p_j$ for every $j \in J_a$. Thus, $p_i \leqslant \neg \bigvee_{j \in J_a} (p_j \wedge \tau_j(a)) = \neg\tau(a)$.

For (3), let $a \in dom^{\mathscr{A}}(\tau) \cap dom^{\mathscr{A}}(\tau_i)$. Since $\tau(a) = \bigvee_{j \in J_a} p_j \wedge \tau_j(a)$, it is enough to show the following two hold.

$$p_i \leqslant \bigwedge_{j \in J_a} (\neg p_j \vee \neg\tau_j(a) \vee \tau_i(a)). \tag{4}$$

$$p_i \leqslant \neg\tau_i(a) \vee \bigvee_{i \in J_a} (p_j \wedge \tau_j(a)). \tag{5}$$

For (4), note that if $i \neq j$, then $p_i \leqslant \neg p_j$; otherwise $\neg\tau_i(a) \vee \tau_i(a) = 1$. For (5), as $i \in J_a$, we have

$$\begin{aligned} p_i &\leqslant \neg\tau_i(a) \vee p_i \\ &= \neg\tau_i(a) \vee (p_i \wedge \tau_i(a)) \\ &\leqslant \neg\tau_i(a) \vee \bigvee_{i \in J_a} (p_j \wedge \tau_j(a)). \end{aligned}$$

■

This completes the proof of the lemma. □

3.3. **Embedding $U^{\mathbb{B}}$ into $\overline{U^{\mathbb{B}}}$.** In this section, we clarify the relationship between $U^{\mathbb{B}}$ and $\overline{U^{\mathbb{B}}}$ by proving that $U^{\mathbb{B}}$ can be elementarily embedded into $\overline{U^{\mathbb{B}}}$. We first define the *hat-names* in $\overline{U^{\mathbb{B}}}$, which is the analog of the *check-names* in $U^{\mathbb{B}}$ (Definition 2.2).

Definition 3.4. For every $x \in U$,

$$\hat{x} = \begin{cases} \{\langle x, 1\rangle\} & x \in \mathscr{A} \\ \{\langle \hat{y}, 1\rangle \mid y \in x\} & x \text{ is a set.} \end{cases}$$

As before, the map $x \mapsto \hat{x}$ preserves Δ_0 assertions from U to $\overline{U^{\mathbb{B}}}$. We note that for any urelement a, $\hat{a}$ is the canonical name of a in $\overline{U^{\mathbb{B}}}$, while $\{\langle a, 1\rangle\}$ is the canonical name of $\{a\}$ in $U^{\mathbb{B}}$. The following lemma demonstrates an expected feature of $\hat{a}$ and will be useful later on.

Lemma 3.7. For every $\tau \in \overline{U^{\mathbb{B}}}$ and $a \in dom^{\mathscr{A}}(\tau)$, $[\![\hat{a} = \tau]\!]^{\overline{U^{\mathbb{B}}}} = \tau(a)$.

Proof. First, $[\![\hat{a} \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} = 1$ and

$$\begin{aligned} [\![\hat{a} \stackrel{\mathscr{A}}{=} \tau]\!]^{\overline{U^{\mathbb{B}}}} &= (\tau(a) \Leftrightarrow 1) \wedge \bigvee_{b \in dom^{\mathscr{A}}(\tau)\setminus\{a\}} \neg\tau(b) \\ &= \tau(a). \end{aligned}$$

Also,

$$\begin{aligned} [\![\tau \subseteq \hat{a}]\!]^{\overline{U^{\mathbb{B}}}} &= \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \Rightarrow [\![\eta \in \hat{a}]\!]^{\overline{U^{\mathbb{B}}}}) \\ &= \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \Rightarrow 0) \\ &= \bigwedge_{\eta \in dom^{\mathbb{B}}(\tau)} \neg\tau(\eta) \\ &\geqslant \tau(a). \end{aligned}$$

Thus, $[\![\hat{a} = \tau]\!]^{\overline{U^{\mathbb{B}}}} = [\![\tau \subseteq \hat{a}]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\hat{a} \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![\hat{a} \stackrel{\mathscr{A}}{=} \tau]\!]^{\overline{U^{\mathbb{B}}}} = \tau(a)$. □

Now we define a natural embedding from $U^{\mathbb{B}}$ to $\overline{U^{\mathbb{B}}}$.

Definition 3.5. For every $\tau \in U^{\mathbb{B}}$,

$$j(\tau) = \begin{cases} \hat{a} & \tau \text{ is some urelement } a \\ \{\langle j(\eta), \tau(\eta)\rangle \mid \eta \in dom(\tau)\} & \tau \text{ is a set.} \end{cases}$$

By a straightforward induction, one can show that j is one-one and hence $j(\tau)$ is a $\mathbb{B}$-valued function; moreover, as $dom(j(\tau)) \subseteq \overline{U^{\mathbb{B}}}$ whenever τ is a set, the incompatibility condition in Definition 3.3 is trivially satisfied, and hence $j(\tau) \in \overline{U^{\mathbb{B}}}$ for every $\tau \in U^{\mathbb{B}}$. We shall write $j\tau$ for $j(\tau)$ from now on.

Lemma 3.8. For every $\tau, \sigma \in U^{\mathbb{B}}$, $[\![\varphi(\tau, \sigma)]\!]^{U^{\mathbb{B}}} = [\![\varphi(j\tau, j\sigma)]\!]^{\overline{U^{\mathbb{B}}}}$, where φ is an atomic formula.

Proof. We prove the lemma by induction on τ and σ.

Suppose that φ is $\mathscr{A}(v_1)$. Then $[\![\mathscr{A}(\tau)]\!]^{U^{\mathbb{B}}} = 1 = [\![\mathscr{A}(j\tau)]\!]^{\overline{U^{\mathbb{B}}}}$ if $\tau \in \mathscr{A}$; otherwise $[\![\mathscr{A}(\tau)]\!]^{U^{\mathbb{B}}} = 0 = [\![\mathscr{A}(j\tau)]\!]^{\overline{U^{\mathbb{B}}}}$.

Suppose that φ is $v_1 \in v_2$. Then

$$\begin{aligned} [\![\tau \in \sigma]\!]^{U^{\mathbb{B}}} &= \bigvee_{\mu \in dom(\sigma)} ([\![\tau = \mu]\!]^{U^{\mathbb{B}}} \wedge \sigma(\mu)) \\ \text{(induction hypothesis)} \quad &= \bigvee_{\mu \in dom(\sigma)} ([\![j\tau = j\mu]\!]^{\overline{U^{\mathbb{B}}}} \wedge j\sigma(j\mu)) \\ &= [\![j\tau \in j\sigma]\!]^{\overline{U^{\mathbb{B}}}}. \end{aligned}$$

Suppose that φ is $v_1 = v_2$. Since j is injective, it suffices to consider the following two cases.

Case 1: τ is some urelement a and $\sigma \neq \tau$. Then $[\![\tau = \sigma]\!]^{U^{\mathbb{B}}} = 0$. If σ is some urelement b, then we have $[\![\hat{a} = \hat{b}]\!]^{\overline{U^{\mathbb{B}}}} = 0$. If σ is a set, then $[\![\hat{a} \overset{\mathscr{A}}{=} j\sigma]\!]^{\overline{U^{\mathbb{B}}}} = 0$ because $j\sigma$ has no urelements in its domain and hence $[\![\hat{a} = j\sigma]\!]^{\overline{U^{\mathbb{B}}}} = 0$.

Case 2: τ and σ are sets. Then $[\![j\tau \overset{\mathscr{A}}{=} j\sigma]\!]^{\overline{U^{\mathbb{B}}}} = 1$. So

$$\begin{aligned} [\![j\tau = j\sigma]\!]^{\overline{U^{\mathbb{B}}}} &= [\![j\tau \subseteq j\sigma]\!]^{\overline{U^{\mathbb{B}}}} \wedge [\![j\sigma \subseteq j\tau]\!]^{\overline{U^{\mathbb{B}}}} \\ &= \bigwedge_{\eta \in dom(\tau)} (\tau(\eta) \Rightarrow [\![j\eta \in j\sigma]\!]^{\overline{U^{\mathbb{B}}}}) \wedge \bigwedge_{\mu \in dom(\sigma)} (\sigma(\mu) \Rightarrow [\![j\mu \in j\tau]\!]^{\overline{U^{\mathbb{B}}}}) \\ &= [\![\tau \subseteq \sigma]\!]^{U^{\mathbb{B}}} \wedge [\![\sigma \subseteq \tau]\!]^{U^{\mathbb{B}}} \\ &= [\![\tau = \sigma]\!]^{U^{\mathbb{B}}}, \end{aligned}$$

where the third line holds by induction hypothesis. □

Unsurprisingly, it is the quantifier case that requires some effort if we wish to show that j is fully elementary. A key lemma will be that every $\tau \in \overline{U^{\mathbb{B}}}$ *mixes* of some mixed-names that are in the range of j with respect to a maximal antichain.

To begin with, observe that for any given $\tau \in \overline{U^{\mathbb{B}}}$, $[\![\neg\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in dom^{\mathscr{A}}(\tau)} \neg\tau(a)$ is the value of τ's being a set. So we may map each τ in $\overline{U^{\mathbb{B}}}$ to its *set counterpart*, τ^{Set}, so that $[\![\tau = \tau^{\text{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = [\![\neg\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}}$ and τ^{Set} lies in the range of j. For example, if $\eta = \{\langle a, p\rangle\}$, then we would obviously want $\eta^{\text{Set}} = \emptyset$; and if $\tau = \{\langle \eta, q\rangle, \langle b, \neg q\rangle\}$, then we would forget about the urelement b and let $\tau^{\text{Set}} = \{\langle \eta^{\text{Set}}, q \wedge \neg p\rangle, \langle \hat{a}, q \wedge p\rangle\}$, making $[\![\tau = \tau^{\text{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = q$. But in general, in the domain of τ there might be different names that have the same set counterpart, which motivates the following definition.

Definition 3.6. For every $\tau \in \overline{U^{\mathbb{B}}}$, τ^{Set} is defined recursively as follows.

1. $dom(\tau^{\text{Set}}) = \{\eta^{\text{Set}} \mid \eta \in dom^{\mathbb{B}}(\tau)\} \cup \{\hat{a} \mid \exists \eta \in dom^{\mathbb{B}}(\tau)\ a \in dom^{\mathscr{A}}(\eta)\}$.
2. For every $\nu \in dom(\tau^{\text{Set}})$,

   if $\nu = \hat{a}$ for some $a \in \mathscr{A}$, then

   $$\tau^{\text{Set}}(\hat{a}) = \bigvee\{\tau(\eta) \wedge \eta(a) \mid a \in dom(\eta) \text{ and } \eta \in dom^{\mathbb{B}}(\tau)\};$$

   if $\nu = \mu^{\text{Set}}$ for some $\mu \in dom^{\mathbb{B}}(\tau)$, then

$$\tau^{\text{Set}}(\mu^{\text{Set}}) = \bigvee\{\tau(\eta) \wedge [\![\neg\mathscr{A}(\eta)]\!]^{\overline{U^{\mathbb{B}}}} \mid \eta \in dom^{\mathbb{B}}(\tau) \text{ and } \eta^{\text{Set}} = \mu^{\text{Set}}\}.$$

Lemma 3.9. For every $\tau \in \overline{U^{\mathbb{B}}}$, there is some $\sigma \in U^{\mathbb{B}}$ such that $j\sigma = \tau^{\mathrm{Set}}$.

Proof. By induction on τ. If $dom(\tau) \subseteq \mathscr{A}$, then $\tau^{\mathrm{Set}} = \emptyset = j(\emptyset)$. Suppose that the lemma holds for every $\eta \in dom^{\mathbb{B}}(\tau)$. Then let

$$\sigma = \{\langle j^{-1}\nu, \tau^{\mathrm{Set}}(\nu)\rangle \mid \nu \in dom(\tau^{\mathrm{Set}})\}.$$

This is well-defined since every $\nu \in dom(\tau^{\mathrm{Set}})$ is either some $\hat{a}$ or η^{Set} for some $\eta \in dom^{\mathbb{B}}(\tau)$. So $\sigma \in U^{\mathbb{B}}$, and it is clear that $j\sigma = \tau^{\mathrm{Set}}$. □

Lemma 3.10. For every $\tau \in \overline{U^{\mathbb{B}}}$, $[\![\tau = \tau^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = [\![\neg\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}}$.

Proof. By induction on the name rank of τ. When $dom(\tau) \subseteq \mathscr{A}$, $\tau^{\mathrm{Set}} = \emptyset$ and the lemma clearly holds. Suppose that the lemma holds for every $\eta \in dom^{\mathbb{B}}(\tau)$. We show that $[\![\tau = \tau^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in dom^{\mathscr{A}}(\tau)} \neg\tau(a)$. First, it is clear that $[\![\tau \overset{\mathscr{A}}{=} \tau^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = \bigwedge_{a \in dom^{\mathscr{A}}(\tau)} \neg\tau(a)$ because τ^{Set} contains no urelements in its domain. The lemma then follows from the next two claims.

Claim 3.10.1. $[\![\tau \subseteq \tau^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = 1$.

Proof of the Claim. We show that for every $\eta \in dom^{\mathbb{B}}(\tau)$, $\tau(\eta) \leqslant [\![\eta \in \tau^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}}$. Fix $\eta \in dom^{\mathbb{B}}(\tau)$. For every $a \in dom^{\mathscr{A}}(\eta)$, $\tau(\eta) \wedge \eta(a) \leqslant \tau^{\mathrm{Set}}(\hat{a})$ and so $\tau(\eta) \wedge \eta(a) \leqslant [\![\eta = \hat{a}]\!]^{\overline{U^{\mathbb{B}}}} \wedge \tau^{\mathrm{Set}}(\hat{a})$ by Lemma 3.7. This shows that

$$\bigvee_{a \in dom^{\mathscr{A}}(\eta)} (\tau(\eta) \wedge \eta(a)) \leqslant \bigvee_{a \in dom^{\mathscr{A}}(\eta)} ([\![\eta = \hat{a}]\!]^{\overline{U^{\mathbb{B}}}} \wedge \tau^{\mathrm{Set}}(\hat{a})). \tag{6}$$

Since

$$\tau(\eta) \wedge [\![\neg\mathscr{A}(\eta)]\!]^{\overline{U^{\mathbb{B}}}} \leqslant \tau^{\mathrm{Set}}(\eta^{\mathrm{Set}})$$

by the definition of τ^{Set}, and

$$[\![\neg\mathscr{A}(\eta)]\!]^{\overline{U^{\mathbb{B}}}} = [\![\eta = \eta^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}}$$

by induction hypothesis, we have

$$\tau(\eta) \wedge [\![\neg\mathscr{A}(\eta)]\!]^{\overline{U^{\mathbb{B}}}} \leqslant [\![\eta = \eta^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} \wedge \tau^{\mathrm{Set}}(\eta^{\mathrm{Set}}). \tag{7}$$

It then follows that

$$\begin{aligned}
\tau(\eta) &= \bigvee_{a \in dom^{\mathscr{A}}(\eta)} (\tau(\eta) \wedge \eta(a)) \vee (\tau(\eta) \wedge [\![\neg\mathscr{A}(\eta)]\!]^{\overline{U^{\mathbb{B}}}}) \\
\text{(by (6) and (7))} \quad &\leqslant \bigvee_{a \in dom^{\mathscr{A}}(\eta)} ([\![\eta = \hat{a}]\!]^{\overline{U^{\mathbb{B}}}} \wedge \tau^{\mathrm{Set}}(\hat{a})) \vee ([\![\eta = \eta^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} \wedge \tau^{\mathrm{Set}}(\eta^{\mathrm{Set}})) \\
&\leqslant \bigvee_{\nu \in dom^{\mathbb{B}}(\tau^{\mathrm{Set}})} ([\![\eta = \nu]\!]^{\overline{U^{\mathbb{B}}}} \wedge \tau^{\mathrm{Set}}(\nu)) \\
&= [\![\eta \in \tau^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}}.
\end{aligned}$$

■

Claim 3.10.2. $[\![\tau^{\mathrm{Set}} \subseteq \tau]\!]^{\overline{U^{\mathbb{B}}}} = 1$.

Proof of the Claim. We show that for every $\nu \in dom(\tau^{\mathrm{Set}})$, $\tau^{\mathrm{Set}}(\nu) \leqslant [\![\nu \in \tau]\!]^{\overline{U^{\mathbb{B}}}}$. Suppose that $\nu = \hat{a}$ for some $a \in dom^{\mathscr{A}}(\mu)$ and $\mu \in dom^{\mathbb{B}}(\tau)$. Let $J_a = \{\eta \in dom^{\mathbb{B}}(\tau) \mid a \in dom(\eta)\}$. Then

$$\begin{aligned}
\tau^{\mathrm{Set}}(\hat{a}) &= \bigvee_{\eta \in J_a} (\tau(\eta) \wedge \eta(a)) \\
\text{(Lemma 3.7)} \quad &= \bigvee_{\eta \in J_a} (\tau(\eta) \wedge [\![\eta = \hat{a}]\!]^{\overline{U^{\mathbb{B}}}}) \\
&\leqslant \bigvee_{\eta \in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \wedge [\![\eta = \hat{a}]\!]^{\overline{U^{\mathbb{B}}}}) \\
&= [\![\hat{a} \in \tau]\!]^{\overline{U^{\mathbb{B}}}}.
\end{aligned}$$

Suppose that $\nu = \mu^{\mathrm{Set}}$ for some $\mu \in dom^{\mathbb{B}}(\tau)$. Let $X_\mu = \{\eta \in dom(\tau) \mid \eta^{\mathrm{Set}} = \mu^{\mathrm{Set}}\}$. For every $\eta \in X_\mu$, $[\![\eta = \mu^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = [\![\eta = \eta^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} = [\![\neg\mathscr{A}(\eta)]\!]^{\overline{U^{\mathbb{B}}}}$ by induction hypothesis. Thus,

$$\begin{aligned}
\tau^{\mathrm{Set}}(\mu^{\mathrm{Set}}) &= \bigvee_{\eta \in X_\mu} (\tau(\eta) \wedge [\![\neg\mathscr{A}(\eta)]\!]^{\overline{U^{\mathbb{B}}}}) \\
&= \bigvee_{\eta \in X_\mu} (\tau(\eta) \wedge [\![\eta = \mu^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}}) \\
&\leqslant \bigvee_{\eta \in dom^{\mathbb{B}}(\tau)} (\tau(\eta) \wedge [\![\eta = \mu^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}}) \\
&= [\![\mu^{\mathrm{Set}} \in \tau]\!]^{\overline{U^{\mathbb{B}}}}.
\end{aligned}$$

■

The lemma is then proved. □

The next lemma is proved by showing that every $\tau \in \overline{U^{\mathbb{B}}}$ mixes $\{\hat{a} \mid a \in dom^{\mathscr{A}}(\tau)\} \cup \{\tau^{\mathrm{Set}}\}$ with respect to the maximal antichain $\{\tau(a) \mid a \in dom^{\mathscr{A}}(\tau)\} \cup \{[\![\neg\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}}\}$.

Lemma 3.11. For every $\tau \in \overline{U^{\mathbb{B}}}$,

$$\bigvee_{\sigma \in U^{\mathbb{B}}} [\![\tau = j\sigma]\!]^{\overline{U^{\mathbb{B}}}} = 1.$$

Proof. Let $X = \{\hat{a} \mid a \in dom^{\mathscr{A}}(\tau)\} \cup \{\tau^{\mathrm{Set}}\}$. For every $\mu \in X$, there is a $\sigma \in U^{\mathbb{B}}$ such that $j\sigma = \mu$ by Lemma 3.9. So we have

$$\begin{aligned}
\bigvee_{\sigma \in U^{\mathbb{B}}} [\![\tau = j\sigma]\!]^{\overline{U^{\mathbb{B}}}} &\geqslant \bigvee_{\mu \in X} [\![\tau = \mu]\!]^{\overline{U^{\mathbb{B}}}} \\
&= \Big(\bigvee_{a \in dom^{\mathscr{A}}(\tau)} [\![\tau = \hat{a}]\!]^{\overline{U^{\mathbb{B}}}}\Big) \vee [\![\tau = \tau^{\mathrm{Set}}]\!]^{\overline{U^{\mathbb{B}}}} \\
\text{(Lemma 3.10)} \quad &= [\![\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} \vee [\![\neg\mathscr{A}(\tau)]\!]^{\overline{U^{\mathbb{B}}}} \\
&= 1.
\end{aligned}$$

□

Theorem 3.12. Let $\mathbb{B}$ be a complete Boolean algebra. Then j, as in Definition 3.5, is an elementary embedding from $U^{\mathbb{B}}$ to $\overline{U^{\mathbb{B}}}$. That is, for any formula $\varphi(v_1, ..., v_n)$ and $\tau_1, ..., \tau_n \in U^{\mathbb{B}}$,

$$\llbracket \varphi(\tau_1, ..., \tau_n) \rrbracket^{U^{\mathbb{B}}} = \llbracket \varphi(j\tau_1, ..., j\tau_n) \rrbracket^{\overline{U^{\mathbb{B}}}}.$$

Proof. The atomic cases are already proved in Lemma 3.8, and the cases for connectives are trivial. So let $\varphi(v_1, ..., v_n)$ be some $\exists v \psi(v, v_1, ..., v_n)$ and $\tau_1, ..., \tau_n \in U^{\mathbb{B}}$. We have

$$\begin{aligned} \llbracket \exists v \psi(v, \tau_1, ..., \tau_n) \rrbracket^{U^{\mathbb{B}}} &= \bigvee_{\sigma \in U^{\mathbb{B}}} \llbracket \psi(\sigma, \tau_1, ..., \tau_n) \rrbracket^{U^{\mathbb{B}}} \\ \text{(induction hypothesis)} \quad &= \bigvee_{\sigma \in U^{\mathbb{B}}} \llbracket \psi(j\sigma, j\tau_1, ..., j\tau_n) \rrbracket^{\overline{U^{\mathbb{B}}}} \\ &\leqslant \llbracket \exists v \psi(v, j\tau_1, ..., j\tau_n) \rrbracket^{\overline{U^{\mathbb{B}}}}. \end{aligned}$$

On the other hand, for every $\tau \in \overline{U^{\mathbb{B}}}$,

$$\begin{aligned} \text{(Lemma 3.11)} \quad \llbracket \psi(\tau, j\tau_1, ..., j\tau_n) \rrbracket^{\overline{U^{\mathbb{B}}}} &= \bigvee_{\sigma \in U^{\mathbb{B}}} \llbracket \tau = j\sigma \rrbracket^{\overline{U^{\mathbb{B}}}} \wedge \llbracket \psi(\tau, j\tau_1, ..., j\tau_n) \rrbracket^{\overline{U^{\mathbb{B}}}} \\ &\leqslant \bigvee_{\sigma \in U^{\mathbb{B}}} \llbracket \psi(j\sigma, j\tau_1, ..., j\tau_n) \rrbracket^{\overline{U^{\mathbb{B}}}} \\ \text{(induction hypothesis)} \quad &= \bigvee_{\sigma \in U^{\mathbb{B}}} \llbracket \psi(\sigma, \tau_1, ..., \tau_n) \rrbracket^{U^{\mathbb{B}}} \\ &= \llbracket \exists v \psi(v, \tau_1, ..., \tau_n) \rrbracket^{U^{\mathbb{B}}}. \end{aligned}$$

Therefore, $\llbracket \exists v \psi(v, j\tau_1, ..., j\tau_n) \rrbracket^{\overline{U^{\mathbb{B}}}} \leqslant \llbracket \exists v \psi(v, \tau_1, ..., \tau_n) \rrbracket^{U^{\mathbb{B}}}$. □

Corollary 3.12.1 (The Fundamental Theorem of $\overline{U^{\mathbb{B}}}$)**.** Assume $\mathrm{ZFU_R}$. Let $\mathbb{B}$ be a complete Boolean-algebra. Then

1. $\overline{U^{\mathbb{B}}} \models \mathrm{ZFU_R}$, and $\overline{U^{\mathbb{B}}} \models$ if $U \models$ AC;
2. $\overline{U^{\mathbb{B}}} \models$ Collection if $U \models$ Collection;
3. $\overline{U^{\mathbb{B}}} \models$ Plenitude if $U \models$ Plenitude;
4. if $U \models$ AC, then $\overline{U^{\mathbb{B}}} \models$ Plenitude only if $U \models$ Plenitude;
5. if $U \models$ AC, then $\overline{U^{\mathbb{B}}} \models$ Tail if $U \models$ Tail.

Proof. By Theorem 2.9 and 3.12. □

3.4. **Fullness and Collection.** Is $\overline{U^{\mathbb{B}}}$ full for every $\mathbb{B}$? It is a standard result (see [6] and [10]) that every Boolean-valued model closed under mixtures is full. However, as $\overline{U^{\mathbb{B}}}$ is a definable class inside U, whether $\overline{U^{\mathbb{B}}}$ is full turns out to depend on what axioms hold in U. The rest of the paper proves the following.

Theorem 3.13. The following are equivalent over $\mathrm{ZFCU_R}$.

1. Collection.
2. $\overline{U^{\mathbb{B}}}$ is full for every complete Boolean algebra $\mathbb{B}$.

The argument for (1) → (2) is standard, and the point is that $\overline{U^{\mathbb{B}}}$ is closed under mixtures. We also note that AC cannot be dropped for proving this direction due to a standard result that AC is equivalent to "$V^{\mathbb{B}}$ is full for every $\mathbb{B}$" over ZF.

To show (1) → (2), assume that Collection holds in U. Fix a complete Boolean-algebra $\mathbb{B}$ and consider $[\![\exists x\varphi(x,\mu)]\!]$[1] for some φ and $\mu \in \overline{U^{\mathbb{B}}}$. Let $q = [\![\exists x\varphi(x,\mu)]\!]$ and let

$$S = \{p \in \mathbb{B} \mid \exists \sigma \in \overline{U^{\mathbb{B}}}\ p \leq [\![\varphi(\sigma,\mu)]\!]\}.$$

By AC, S has a maximal antichain $\{p_i : i \in I\}$ which is below q. By Collection, there is a set v such that for every p_i, there is a mixed $\mathbb{B}$-name $\sigma \in v$ with $p_i \leq [\![\varphi(\sigma,\mu)]\!]$. So we can use AC to choose a $\tau_i \in \overline{U^{\mathbb{B}}}$ for each $i \in I$ such that $p_i \leq [\![\varphi(\tau_i,\mu)]\!]$. By Lemma 3.6, there is a $\tau \in \overline{U^{\mathbb{B}}}$ such that $p_i \leq [\![\tau = \tau_i]\!]$ for every $i \in I$. So $p_i \leq [\![\varphi(\tau,\mu)]\!]$ for every i. Since $\bigvee_{i\in I} p_i = q$ it follows that $[\![\exists x\varphi(x,\mu)]\!] = [\![\varphi(\tau,\mu)]\!]$. Hence, $\overline{U^{\mathbb{B}}}$ is full.

Now we show that the fullness of every $\overline{U^{\mathbb{B}}}$ implies Collection, and the argument does not require AC. We shall first prove a standard fact regarding the powerset algebra $P(I)$ for any given set I, where $\leqslant$ is $\subseteq$.

Lemma 3.14. Let $\mathbb{B} = P(I)$ for some set I. Then for every $\tau \in \overline{U^{\mathbb{B}}}$,

$$\bigvee_{x \in V(ker(\tau \cup \mathbb{B}))} [\![\tau = \hat{x}]\!] = 1.$$

Proof. We prove it by induction on τ. Suppose that the lemma holds for every $\eta \in dom^{\mathbb{B}}(\tau)$. Because $P(I)$ is an atomic Boolean algebra, for every $\eta \in dom^{\mathbb{B}}(\tau)$ and $i \in I$, there is a unique $v^i_\eta \in V(ker(\eta \cup \mathbb{B}))$ such that $\{i\} \leqslant [\![\eta = \hat{v^i_\eta}]\!]$. So for any $i \in I$, we define

$$x_i = \begin{cases} a & a \in dom^{\mathscr{A}}(\tau) \text{ and } \{i\} \leqslant \tau(a) \\ \{v^i_\eta \mid i \in \tau(\eta) \text{ and } \eta \in dom^{\mathbb{B}}(\tau)\} & \text{otherwise.} \end{cases}$$

The incompatibility condition of $U^{\mathbb{B}}$-names ensures that x_i is well-defined and $ker(x_i) \subseteq ker(\tau \cup \mathbb{B})$ for each $i \in I$. We show that τ mixes $\{\hat{x}_i \mid i \in I\}$ with respect to $\{\{i\} \mid i \in I\}$. Fix an $i \in I$.

Claim 3.14.1. $\{i\} \leqslant [\![\tau \subseteq \hat{x}_i]\!]$.

Proof of the Claim. Let $\eta \in dom^{\mathbb{B}}(\tau)$. If $\{i\} \not\leqslant \tau(\eta)$, then $\{i\} \leqslant \neg\tau(\eta) \leqslant \tau(\eta) \Rightarrow [\![\eta \in \hat{x}_i]\!]$. If $i \in \tau(\eta)$, then $v^i_\eta \in x_i$, and hence $\{i\} \leqslant [\![\eta = \hat{v^i_\eta}]\!] \leqslant [\![\eta \in \hat{x}_i]\!] \leqslant \tau(\eta) \Rightarrow [\![\eta \in \hat{x}_i]\!]$. ■

Claim 3.14.2. $\{i\} \leqslant [\![\hat{x}_i \subseteq \tau]\!]$.

Proof of the Claim. Note that $[\![\hat{x}_i \subseteq \tau]\!] = \bigwedge_{v^i_\eta \in x_i} [\![\hat{v^i_\eta} \in \tau]\!]$. Let $v^i_\eta \in x_i$. Then $\{i\} \leqslant \tau(\eta)$. So $\{i\} \leqslant \tau(\eta) \wedge [\![\eta = \hat{v^i_\eta}]\!] \leqslant [\![\hat{v^i_\eta} \in \tau]\!]$. ■

Claim 3.14.3. $\{i\} \leqslant [\![\tau \overset{\mathscr{A}}{=} \hat{x}_i]\!]$.

Proof of the Claim. If $\hat{x}_i$ is some $\hat{a}$, where $a \in dom^{\mathscr{A}}(\tau)$, then $\{i\} \leqslant \tau(a) \leqslant [\![\tau \overset{\mathscr{A}}{=} \hat{x}_i]\!]$ (Lemma 3.7). Otherwise, $dom^{\mathscr{A}}(\hat{x}_i) = \emptyset$ so $[\![\tau \overset{\mathscr{A}}{=} \hat{x}_i]\!] = \bigwedge_{a \in dom^{\mathscr{A}}(\tau)} \neg\tau(a)$; and by the definition of x_i, $\{i\} \leqslant \neg\tau(a)$ for every $a \in dom^{\mathscr{A}}(\tau)$. ■

Therefore, we have $\{i\} \leqslant [\![\tau = \hat{x}_i]\!]$ for every $i \in I$ and so $\bigvee_{i\in I} [\![\tau = \hat{x}_i]\!] = 1$. This proves the lemma. □

[1]The superscript will be omitted from now on since we will only work in $\overline{U^{\mathbb{B}}}$.

Lemma 3.15. Let $\mathbb{B} = P(I)$ for some set I. For every $x_1, ..., x_n$,

(1) $\varphi(x_1, ..., x_n) \leftrightarrow [\![\varphi(\hat{x_1}, ..., \hat{x_n})]\!] = 1$;
(2) $\neg\varphi(x_1, ..., x_n) \leftrightarrow [\![\varphi(\hat{x_1}, ..., \hat{x_n})]\!] = 0$.

Proof. By induction on the complexity of φ. It remains to prove the quantifier case. Let $\varphi(v_1, ..., v_n) = \exists v\psi(v, v_1, ..., v_n)$. Suppose that $\exists v\psi(v, x_1, ..., x_n)$. Then $[\![\exists v\psi(v, \hat{x_1}, ..., \hat{x_n})]\!] = 1$ by induction hypothesis. Suppose that $\neg\exists v\psi(v, x_1, ..., x_n)$. Then by induction hypothesis, $[\![\psi(\hat{x}, \hat{x_1}, ..., \hat{x_n})]\!] = 0$ for every x. And for every $\tau \in \overline{U^{\mathbb{B}}}$,

$$\begin{aligned}
[\![\psi(\tau, \hat{x_1}, ..., \hat{x_n})]\!] &= [\![\psi(\tau, \hat{x_1}, ..., \hat{x_n})]\!] \wedge \bigvee_{x \in U} [\![\tau = \hat{x}]\!] \qquad \text{(Lemma 3.14)} \\
&\leqslant \bigvee_{x \in U} [\![\psi(\hat{x}, \hat{x_1}, ..., \hat{x_n})]\!] \\
&= 0.
\end{aligned}$$

Hence $[\![\exists v\psi(v, x_1, ..., x_n)]\!] = 0$ and the lemma follows. □

Proof of Theorem 3.13. (2) → (1). Assume that for every $\mathbb{B}$, $\overline{U^{\mathbb{B}}}$ is full. Suppose that in U, $\forall x \in u \exists y \varphi(x, y)$ for some u. We wish to find some set A of urelements such that $\forall x \in u \exists y \in V(A)\varphi(x, y)$. This will suffice for Collection because since u is a set, we can find a large enough α such that for all $x \in u$, there is some $y \in V_\alpha(A)$ with $\varphi(x, y)$.

Let $\mathbb{B} = \mathcal{P}(u)$. By Lemma 3.15, it follows that $\overline{U^{\mathbb{B}}} \models \forall x \in \hat{u} \exists y \varphi(x, y)$. By Lemma 3.6, there is some τ mixing $\{\hat{x} \mid x \in u\}$ with respect to the antichain $\{\{x\} \mid x \in u\}$. That is, $\{x\} \leqslant [\![\tau = \hat{x}]\!]$ for every $x \in u$. So $[\![\tau \in \hat{u}]\!] = \bigvee_{x \in u} [\![\tau = \hat{x}]\!] = 1$ and hence $[\![\exists y \varphi(\tau, y)]\!] = 1$. Since $\overline{U^{\mathbb{B}}}$ is full, there is some $\sigma \in \overline{U^{\mathbb{B}}}$ with $[\![\varphi(\tau, \sigma)]\!] = 1$. Let $A = ker(\mathbb{B} \cup \sigma)$. By Lemma 3.14, $\bigvee_{y \in V(A)} [\![\sigma = \hat{y}]\!] = 1$.

Let $x \in u$. Then there is some $y \in V(A)$ such that $\{x\} \leqslant [\![\sigma = \hat{y}]\!]$. So $\{x\} \leqslant [\![\sigma = \hat{y} \wedge \tau = \hat{x} \wedge \varphi(\tau, \sigma)]\!]$ and so $\{x\} \leqslant [\![\varphi(\hat{x}, \hat{y})]\!]$. By Lemma 3.15, it follows that $[\![\varphi(\hat{x}, \hat{y})]\!]$ must be 1 and hence $\varphi(x, y)$. This shows that $\forall x \in u \exists y \in V(A)\varphi(x, y)$, which completes the proof. □

References

[1] Hisashi Aratake. "Sheaves of structures, Heyting-valued structures, and a generalization of Łoś's theorem". In: *Mathematical Logic Quarterly* 67.4 (2021), pp. 445–468.

[2] John L. Bell. *Set Theory : Boolean-Valued Models and Independence Proofs: Boolean-Valued Models and Independence Proofs*. Oxford University Press UK, 2005.

[3] Andreas Blass and Andrej Ščedrov. *Freyd's models for the independence of the axiom of choice*. Vol. 404. American Mathematical Soc., 1989.

[4] Eric J. Hall. "A Characterization of Permutation Models in Terms of Forcing". In: *Notre Dame Journal of Formal Logic* 43.3 (2002), pp. 157–168. DOI: 10.1305/ndjfl/1074290714.

[5] Eric J. Hall. "Permutation Models and SVC". In: *Notre Dame Journal of Formal Logic* 48.2 (2007), pp. 229–235. DOI: 10.1305/ndjfl/1179323265.

[6] Joel David Hamkins and Daniel Evan Seabold. "Well-founded Boolean ultrapowers as large cardinal embeddings". In: *arXiv preprint arXiv:1206.6075* (2012).
[7] Thomas Jech. *Set theory*. Springer Science & Business Media, 2013.
[8] Moreno Pierobon and Matteo Viale. "Boolean valued models, presheaves, and étalé spaces". In: *arXiv preprint arXiv:2006.14852* (2020).
[9] Andrea Vaccaro and Matteo Viale. "Generic absoluteness and boolean names for elements of a Polish space". In: *Bollettino dell'Unione Matematica Italiana* 10.3 (2017), pp. 293–319.
[10] Matteo Viale. "Notes on forcing". Manuscript. 2014.
[11] Xinhe Wu. "Boolean-valued models and their applications". PhD thesis. Massachusetts Institute of Technology, 2022.
[12] Bokai Yao. *Forcing with Urelements*. 2023. arXiv: `2212.13627 [math.LO]`.
[13] Bokai Yao. "Set Theory with Urelements". PhD thesis. University of Notre Dame, 2023. arXiv: `2303.14274`.