ABSTRACTION PRINCIPLES AND SIZE OF REALITY

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ABSTRACT. The Fregean ontology can be naturally interpreted within set theory with urelements, where objects are interpreted as sets and urelements, and concepts as classes. We investigate how the size of reality, i.e., the number of urelements, affects the Fregean abstraction principles. First, based on recent work by Hamkins [7], we show that, in certain natural urelement set theories, Basic Law V is equivalent to the urelements forming a set. Second, we construct natural models of urelement set theory in which Hume's Principle fails for sets. Finally, assuming the consistency of an inaccessible cardinal, we produce a model of second-order urelement set theory with a global well-ordering, where Hume's Principle fails.

1. INTRODUCTION

Fregean abstractions principles aim to establish a certain correspondence between objects and concepts. Frege's *Basic Law V* asserts that each concept is associated with an object, called its *extension*, such that two concepts are co-extensional if and only if they have the same extension. Another prominent instance of Fregean abstraction is *Hume's Principle*, according to which every concept is associated with an object, called its *number*, such that two concepts are equinumerous if and only if they have the same number. Although Frege's system is inconsistent due to Russell's paradox, the neo-logicist project has succeeded in using consistent fragments of Frege's system to recover arithmetic. On the other hand, the Zermelo-Fraenkel set theory, as a different framework from Frege's, provides a unified foundation for mathematics. It is thus natural to investigate the relationship between these two foundational frameworks.

The relationship between abstraction principles and set theory has been studied extensively in the literature. Most of these studies focus on two issues. One is the examination of the model-theoretic properties of theories of abstraction principles using ZFC as a meta-theory (e.g., see Fine [4], Shapiro [14], and Shapiro & Roberts [12]). The other is the attempt to develop set theory itself through abstraction principles (e.g., see Boolos [2], Shapiro & Weir [15], and Shapiro & Uzquiano [13]). However, there is another perspective to consider: how do abstraction principles behave if we formulate them as set-theoretic principles?

To begin with, the Fregean framework can be naturally interpreted in the iterative conception of set: at the initial stage we have some basic objects called *urelements*, i.e., *non-sets* that are members form sets, and then there are sets of these urelements, sets of sets of them, and so on. In this picture, Fregean concepts can be seen as *classes* of the objects formed in the iterative procedure, and the concepts that are too big to coincide with sets are the *proper classes*. The connection between the Fregean framework and set theory extends beyond this analogy: recent work by Hamkins [7] shows that ZF provides a *deflationary* account of

Fregean abstractions principles. In particular, Hamkins shows that every model of ZF equipped with definable classes has a second-order definable map fulfilling Basic Law V ([7, Theorem 1]) as well as a second-order definable map fulfilling Hume's Principle ([7, Theorem 7]).

However, ZF set theory is not the most natural set-theoretic framework for studying abstraction principles because it excludes urelements—every object in ZF is assumed to be a set. Yet, abstraction principles should be *universally applicable* (Shapiro [14] and Shapiro & Roberts [12]), meaning they are intended to talk about *all* concepts along with *all* objects. Therefore, it is more natural to consider abstraction principles within *set theory with urelements*, a framework that allows all kinds of objects to be members of sets. A natural question immediately arises: Is ZF set theory *with urelements* still able to provide a deflationary account of Fregean abstraction principles?

In this paper, we show that a deflationary account of abstraction principles in urelement set theory turns out to be contingent upon the size of reality, i.e., how many urelements there are. For example, in the models of some suitable urelement set theories, a deflationary account of Basic Law V is available if and only if the urelements form a set (Theorem 3 and 4); when there are unboundedly many urelements, no deflationary account of Basic Law V is possible (Theorem 5). In the case of Hume's Principle, Gauntt[5] and Lévy [10] independently showed that $\mathrm{ZFU_R}$ has models with proper-class many urelements where cardinality for sets is not definable; in other words, in these models, there is no definable map fulfilling Hume's Principle for sets. We improve upon their result by constructing models of $\mathrm{ZFU_R}$ + Reflection Principle + DC_κ-scheme, for any given infinite cardinal κ, where Hume's Principle fails for sets (Theorem 7). Furthermore, we construct a model of Kelley-Morse class theory that has a global well-ordering but no definable map fulfilling Hume's Principle (Theorem 12). The philosophical implications of these results on Hume's Principle are discussed at the end.

2. Preliminaries on urelement set theory

The language of urelement set theory, in addition to $\in$, contains a unary predicate $\mathscr{A}$ for urelements. The axioms of $\mathrm{ZFU_R}$ (R for Replacement) include Foundation, Pairing, Union, Powerset, Infinity, Separation, Replacement, Extensionality *for sets*, and the axiom that no urelements have members (see [17, Section 1.2] for the precise formulation of these axioms). $\mathrm{ZFCU_R}$ is $\mathrm{ZFU_R}$ + the Axiom of Choice. Note that $\mathrm{ZFCU_R}$ allows a proper class of urelements.

Although $\mathrm{ZFCU_R}$ looks very much like *the* urelement analog of ZFC, there is, in fact, a hierarchy of axioms that are independent from $\mathrm{ZFCU_R}$ ([17, Theorem 17]), some of which are ZF-theorems. For instance, it is folklore that $\mathrm{ZFCU_R}$ cannot prove the Collection Principle and the Reflection Principle.

(Collection) $\forall w, u(\forall x \in w\ \exists y \varphi(x, y, u) \to \exists v \forall x \in w\ \exists y \in v\ \varphi(x, y, u))$.

(RP) For every set x there is a transitive set t with $x \subseteq t$ such that for every $x_1, ..., x_n \in t$, $\varphi(x_1, ..., x_n) \leftrightarrow \varphi^t(x_1, ..., x_n)$.

This is because $\mathrm{ZFCU_R}$ has models where the urelements form a proper class but every set of urelements is finite (see e.g., [17, Theorem 27 (3)])—call these models *finite-kernel models*. In *finite-kernel* models, for every $n \in \omega$, there is a set of urelements of size n, but there is no corresponding collection set. In the subsequent sections we will introduce some more ZFC-theorems that are not provable over

$\mathrm{ZFCU_R}$. Thus, $\mathrm{ZF(C)U_R}$ is a rather weak set theory, and the subscript is used to emphasizing this point.

Now we review some basic notations and facts about $\mathrm{ZFU_R}$. Every object x in $\mathrm{ZFU_R}$ has a *kernel*, denoted by $ker(x)$, which is the set of the urelements in the transitive closure of $\{x\}$. A set is pure if its kernel is empty. V denotes the class of all pure sets. Ord is the class of all ordinals, which are transitive *pure* sets well-ordered by the membership relation. $\mathscr{A}$ will also stand for the class of all urelements. $A \subseteq \mathscr{A}$ thus means "A is a set of urelements". For any $A \subseteq \mathscr{A}$, the $V_\alpha(A)$-hierarchy is defined as usual, i.e.,

$V_0(A) = A$;
$V_{\alpha+1}(A) = P(V_\alpha(A)) \cup A$;
$V_\gamma(A) = \bigcup_{\alpha<\gamma} V_\alpha(A)$, where γ is a limit;
$V(A) = \bigcup_{\alpha \in Ord} V_\alpha(A)$.

For every x and set $A \subseteq \mathscr{A}$, $x \in V(A)$ if and only if $ker(x) \subseteq A$. U denotes the class of all objects, i.e., $U = \bigcup_{A \subseteq \mathscr{A}} V(A)$. The rank of an object x, denoted by $\rho(x)$, is the least ordinal α such that $x \in V_\alpha(A)$ for some $A \subseteq \mathscr{A}$. When there is only a set of urelements, for every α the objects of rank α form a set. An important feature of $\mathrm{ZFU_R}$ is that U has many non-trivial automorphisms. Every permutation π of a set of urelements can be extended to a definable permutation of $\mathscr{A}$ by letting π be identity elsewhere, and π can be further extended to a permutation of U by letting πx be $\{\pi y : y \in x\}$ for every set x. Such π preserves $\in$ and thus is an automorphism of U. For every x and automorphism π, π point-wise fixes x whenever π point-wise fixes $ker(x)$.

We will also discuss *class theories* with urelements. For instance, every model U of $\mathrm{ZFU_R}$ equipped with definable classes will be a model of *Gödel-Benarys class theory* (GB) with urelements, while *Kelley-Morse class theory* (KM) with urelements is the stronger theory which includes the impredicative comprehension scheme. In the absence of a global well-ordering, different axiomatizations of urelement class theory also come apart (see [17, Section 4.1]), but we shall not be concerned with these subtleties in this paper. In class theory, every permutation π of a set of urelements can also be extended to a permutation of classes that preserves the second-order assertions by letting $\pi X = \{\pi x : x \in X\}$ for every class X.

3. Basic law V in urelement set theory

A map $X \mapsto \epsilon X$ is said to fulfill Basic Law V if for each class X, ϵX is a first-order object and for any classes X and Y, $\epsilon X = \epsilon Y \leftrightarrow \forall x(x \in X \leftrightarrow x \in Y)$. Such a map is also said to be an *extension-assignment* map. Hamkins [7] shows that in every model of ZF equipped with definable classes, there is a second-order definable extension-assignment map.[1] One key step of Hamkins' proofs uses *Scott's trick*, namely, every class can be represented by the *set* of its elements with minimal rank. As we noted before, Scott's trick is available as long as the urelements form a set.

[1]Hamkins [7, Section 6] also explains how one should reconcile his result with Russell's paradox. Basically, Russell's paradox proves the inconsistency between Basic Law V and a certain naive comprehension principle for concepts. In the context of set theory, Russell's paradox can be viewed as the fact that no model Kelley-Morse class theory has a definable map fulfilling Basic Law V.

Theorem 1 ([7, Thereom 1])**.** Let U be a model of $\mathrm{ZFU_R}$ + "$\mathscr{A}$ is a set" equipped with definable classes. Then U has a definable map that fulfills Basic Law V.

Proof. We first in the meta-theory fix a enumeration $\psi_0, ..., \psi_n, ...$ of the formulas of urelement set theory, and this enumeration will be the standard part of a definable enumeration in U. Furthermore, for every standard natural number k, $\mathrm{ZFU_R}$ has a definable Σ_k-truth predicate. Thus, given a definable class X of U, we let $\varphi(X, \epsilon X)$ be the second-order assertion

> "ϵX is an ordered pair $\langle \ulcorner \psi_n \urcorner, u\rangle$, where ψ_n is a Σ_k formula for some k and u is the set of parameters p with minimal rank such that there is a Σ_k truth predicate T with $\forall x(x \in X \leftrightarrow T(\ulcorner \psi_n \urcorner, \langle x, p\rangle))$, and no preceding formula ψ_i has this property."

It then easy to check that the map $X \mapsto \epsilon X$ then fulfills Basic Law V (see [7, page 6]). □

The definability of the extension-assignment map $X \mapsto \epsilon X$, as Hamkins notes, is a major difference between his result and the earlier consistency proofs of Basic Law V given by Parson [11], Bell [1] and Burgess [3]. This definability feature is also philosophically intereseting since it provides a *identity criteria* for extensions as objects, which is analogous to a solution to the famous Julius Caesar Problem (see [7, Section 5] for more on this).

However, in urelement set theory, a deflationary account of Basic Law V is not necessarily available.

Lemma 2. Let U be a model of $\mathrm{ZFU_R}$ in which the following holds.

> (*) For every $A \subseteq \mathscr{A}$, there is a countably infinite $B \subseteq \mathscr{A}$ such that $B \cap A = \emptyset$.

Then U, equipped with definable classes, has no parametrically definable map that fulfills Basic Law V.

Proof. Suppose *for reductio* that in U some second-order formula $\varphi(X, \epsilon X, P)$ with a parameter P defines a extension-assignment map such that for every definable classes X, Y of U,

$$\epsilon X = \epsilon Y \leftrightarrow \forall x(x \in X \leftrightarrow x \in Y).$$

P is a definable class of U so $P = \{x \in U : U \models \psi(x, z)\}$ for some first-order formula ψ and $z \in U$. In U, let $A \subseteq \mathscr{A}$ be an infinite set of urelements disjoint from $ker(z)$ and define $X = \{B \subseteq \mathscr{A} : B - A \text{ is finite}\}$.

Claim 2.1. $A \cap ker(\epsilon X)$ is not empty.

Proof of the Claim. Suppose otherwise. Let $A' \subseteq \mathscr{A}$ be such that $A' \sim \omega$ and $A' \cap (A \cup ker(\epsilon X) \cup ker(z)) = \emptyset$, which exists by (*). Let A_1 be a countably infinite subset of A and π be an automorphism that swaps A_1 and A' while point-wise fixing everything else (in particular, ϵX and P). Since $\varphi(X, \epsilon X, P)$, $\varphi(\pi X, \pi\epsilon X, \pi P)$ and so $\varphi(\pi X, \epsilon X, P)$. Thus, πX and X must be co-extensional. But $\pi X = \{B \subseteq \mathscr{A} : B - \pi A \text{ is finite}\}$, and $A_1 \in X$ but $A_1 \notin \pi X$–contradiction. ■

Fix some urelements $a \in ker(\epsilon X) \cap A$ and $b \notin ker(\epsilon X) \cup ker(z)$. Let σ be the automorphism that only swaps a and b. Consequently, $\sigma\epsilon X \neq \epsilon X$. As X and σX are co-extensional, it follows that $\epsilon\sigma X = \epsilon X$. Moreover, $\varphi(\sigma X, \sigma\epsilon X, P)$ so $\sigma\epsilon X = \epsilon\sigma X = \epsilon X$—contradiction. □

The Axiom of Countable Choice (AC_ω), which states that every countable family of non-empty sets admit a choice function, implies that every infinite set has an countably infinite subset.

Theorem 3. Let U be a model of ZFU_R + Collection + AC_ω equipped with definable classes. The following are equivalent.

(1) The urelements in U form a set.
(2) U has a definable map that fulfills Basic Law V.
(3) U has a parametrically definable map that fulfills Basic Law V.

Proof. (1) $\to$ (2) follows from Theorem 1. It remains to prove (3) $\to$ (1). By Lemma 2, it suffices to show that over ZFU_R + Collection + AC_ω, the principle (*) in Lemma 2 holds if the urelements do not form a set. So suppose that $\mathscr{A}$ is a proper class and consider any $A \subseteq \mathscr{A}$. Since we have for every $n \in \omega$, there is a $B \subseteq \mathscr{A}$ with $B \cap A = \emptyset$ and $|B| = n$, it follows from Collection that there is an infinite $C \subseteq \mathscr{A}$ that is disjoint from A, which has a countably infinite subset by AC_ω. □

The use of Collection can be avoided if we assume a stronger choice principle. The Axiom of Dependent Choice (DC), which is stronger than AC_ω, states that for every relation on a set without terminal nodes, there is an infinite sequence threading the relation. The *Dependence Choice Scheme* is a class version of the DC.

> (DC-scheme) If for every x there is some y such that $\varphi(x, y, u)$, then for every p there is an infinite sequence s such that $s(0) = p$ and $\varphi(s(n), s(n+1), u)$ for every $n < \omega$.

It is known that $ZFCU_R$ + DC-scheme does not prove Collection [17, Theorem 17].

Theorem 4. Let U be a model of ZFU_R + DC-scheme equipped with definable classes. The follow are equivalent.

(1) The urelements in U form a set.
(2) U has a definable map that fulfills Basic Law V.
(3) U has a parametrically definable map that fulfills Basic Law V. □

Proof. Again, we show that (3) $\to$ (1) by observing that over ZFU_R + DC-scheme the principle (*) in Lemma 2 holds if the urelements do not form a set. Assume that $\mathscr{A}$ is a proper class and let $A \subseteq \mathscr{A}$. Since for every x there is some y such that $ker(x) \subsetneq ker(y)$ and $ker(y) - (A \cup ker(x)) \neq \emptyset$, by the DC-scheme there is an infinite sequence $\langle s_n : n < \omega \rangle$ such that $ker(s_n) \subsetneq ker(s_{n+1})$ and $ker(s_{n+1}) - (A \cup ker(s_n)) \neq \emptyset$ for every $n < \omega$. So $ker(s) - A$ is an infinite set of urelements disjoint from A, which has a countably infinite subset by the DC-scheme. □

Note that the argument above also shows that $ZFCU_R$ does not prove the DC-scheme, because (*) fails in the finite-kernel model.

Plenitude is the axiom that for every well-ordered cardinal κ, there is a set of urelements of size κ. Next, we show that Plenitude is inconsistent with Basic Law V without using any choice principle. We also note that over ZFU_R, Plenitude does not imply either Collection or the DC-scheme (although it implies both of them over $ZFCU_R$, see [17, Theorem 17 and Theorem 36]).

Theorem 5. No model of ZFU_R + Plenitude, equipped with definable classes, has a parametrically definable map that fulfills Basic Law V.

Proof. Suppose *for reductio* that in some model U of $\mathrm{ZFU_R}$ + Plenitude we can define an extension-assignment map ϵ using some parameter P defined by some first-order object z in U. Note that as in ZF, $\mathrm{ZFU_R}$ proves that every set x has a *Hartogs number*, $\aleph(x)$, which is the least ordinal that does not inject into x. Let $\kappa = \aleph(ker(z))$ and fix some set of urelements C of size κ. It follows that C must have a subset A of size κ that is disjoint from $ker(z)$. We can then find some $D \subseteq \mathscr{A}$ of size κ^+, which will have a subset A' of size κ that is disjoint from $A \cup ker(z)$. Define $X = \{B \subseteq \mathscr{A} : B - A \text{ is finite}\}$. As in the proof of Claim 2.1, $ker(\epsilon X) \cap A$ cannot be empty since otherwise we can swap A and A' to get a contradiction. But then there is an automorphism σ that swaps some urelement $a \in ker(\epsilon X) \cap A$ with some urelement $b \notin ker(\epsilon X) \cup ker(z)$ while fixing P. Consequently, $\sigma\epsilon X \neq \epsilon X$, and $\epsilon\sigma X = \epsilon X$ because X and σX are co-extensional. However, σ fixes P so $\sigma\epsilon X = \epsilon\sigma X$—contradiction. □

While Russell's paradox shows that Basic Law V is inconsistent with the impredicative comprehension, the theorems above reveal the tensions between Basic Law V and the number of urelements. In other words, in the context of urelement set theory Basic Law V can be seen as having a specific ontological commitment.

4. Hume's Principle in urelement set theory

4.1. **First-Order Hume's Principle.** A map $x \mapsto \#x$ fulfills Hume's Principle for *sets* if it maps each set to some first-order object $\#x$ such that two sets x and y are equinumerous just in case $\#x = \#y$. The existence of a map as such amounts to the definability of cardinality for sets. With AC, cardinality is definable since we can map each set x to the least ordinal to which x is equinumerous. More generally, whenever the urelements form a set, for every set x we can again use Scott's trick and let $\#x = \{y : y \sim x \wedge \forall z(z \sim x \rightarrow \rho(z) \leq \rho(y))\}$. Cardinality, however, is not always definable when a proper class of urelements is allowed.

Theorem 6 (Gauntt [5]; Lévy [10])**.** There are models of $\mathrm{ZFU_R}$ which have no parametrically definable map that fulfills Hume's Principle for sets. □

Since $\mathrm{ZFU_R}$, as we noted earlier, is a rather weak theory, two natural questions arise. First, can $\mathrm{ZFU_R}$ prove Hume's Principle if it is complemented with some ZF-theorems such as the Reflection Principle? Second, does any weaker choice principle imply Hume's Principle? To clarify the second question, let us review the DC_κ-hierarchy. For every infinite well-ordered cardinal κ, DC_κ is the following axiom generalizing DC.

> (DC_κ) For every set x and relation $r \subseteq x^{<\kappa} \times x$, if for every $s \in x^{<\kappa}$, there is some $w \in x$ such that $\langle s, w \rangle \in r$, then there is an $f : \kappa \to x$ such that $\langle f \restriction \alpha, f(\alpha) \rangle \in r$ for all $\alpha < \kappa$.

The DC_κ-scheme is the class version of DC_κ.

> (DC_κ-scheme) If for every x there is some y such that $\varphi(x, y, u)$, then there is some function $f : \kappa \to U$ such that $\varphi(f \restriction \alpha, f(\alpha), u)$ for every $\alpha < \kappa$.

Since $\mathrm{ZFCU_R}$ proves DC_κ for every κ, by a standard argument $\mathrm{ZFCU_R}$ + $\mathscr{A}$ is a set proves that the DC_κ-scheme for every κ ([17, Lemma 23]). One can verify that the DC_ω-scheme is indeed a reformulation of the DC-scheme and that the DC_κ-scheme is equivalent to the following ([17, Proposition 13]).

For every definable class X, if for every $s \in X^{<\kappa}$ there is some $y \in X$ with $\varphi(x, y, u)$, then there is some function $f : \kappa \to X$ such that $\varphi(f \restriction \alpha, f(\alpha), u)$ for every $\alpha < \kappa$.

We will show that no DC_κ-scheme by itself implies Hume's Principle for sets even if we also assume the Reflection Principle, which strengthens Gauntt and Lévy's Theorem.

Theorem 7. Let κ be any infinite cardinal. There is a model of $\mathrm{ZFU_R}$ + RP + DC_κ-scheme which have no parametrically definable map that fulfills Hume's Principle for sets.

Proof. The first step is to build a *permutation model* that violates AC. We assume some familiarity with permutation models and only provide some necessary details of the construction. For a detailed presentation of this topic, see [9]. Fix some infinite cardinal κ. We start with a model U of $\mathrm{ZFCU_R}$ in which $\mathscr{A}$ is a set of size κ^+. Enumerate $\mathscr{A}$ with $\kappa^+ \times \kappa^+$, i.e., $\mathscr{A} = \bigcup_{\alpha<\kappa^+} A_\alpha$, where each row A_α has size κ^+. Let $\mathcal{G}$ be the group of permutations of $\mathscr{A}$ such that for every $\pi \in \mathcal{G}$ and $\alpha < \kappa^+$, $\pi A_\alpha = A_\alpha$. Let $I = \{E \subseteq \mathscr{A} : |E \cap A_\alpha| < \kappa^+ \text{ for each } \alpha < \kappa^+\}$ be an ideal of $\mathscr{A}$. For every x, define $sym(x) = \{\pi \in \mathcal{G}_A : \pi x = x\}$; if x is a set, define $fix(x) = \{\pi \in \mathcal{G}_A : \pi y = y \text{ for all } y \in x\}$. We say an object is *symmetric* if there some $E \in I$, called a *support* of x, such that $fix(E) \subseteq sym(x)$. The permutation model W is the class of all *hereditarily symmetric* objects, i.e., $W = \{x \in U : x \text{ is symmetric} \land x \subseteq W\}$, which is a model of $\mathrm{ZFU_R}$ (see [9, Theorem 4.1], or [17, Theorem 33] for a more general proof).

Note that $\mathscr{A} \in W$ so there is only a set of urelements in W and consequently Hume's Principle still holds in W. The next step is to produce a model with proper-class many urelements in which Hume's Principle fails. In W, define a class of sets $\mathscr{I}$ of urelements such that

$$\mathscr{I} = \{B \subseteq \mathscr{A} : B \text{ is a subset of } \kappa\text{-many } A_\alpha\}.$$

Let $W^{\mathscr{I}} = \{x \in W : ker(x) \in \mathscr{I}\}$. $\mathscr{I}$ is a class *ideal* of $\mathscr{A}$ which picks out some "small" sets of urelements, while $W^{\mathscr{I}}$ is the class of all objects in W whose kernel is small in the sense of $\mathscr{I}$. By [17, Theorem 26], $W^{\mathscr{I}} \models \mathrm{ZFU_R}$ and there is a proper class of urelements in $W^{\mathscr{I}}$.

Lemma 8. $W^{\mathscr{I}} \models \mathrm{DC}_\kappa$-scheme.

Proof. First observe that $W^{\mathscr{I}}$ is closed-under κ-sequences. Consider any $s : \kappa \to W^{\mathscr{I}}$. In U, for each $\alpha < \kappa$ choose some $E_\alpha \in I$ to be a support of s_α. Then $\bigcup_{\alpha<\kappa} E_\alpha \in I$, which is a support of s. And since $ker(s) = \bigcup_{\alpha<\kappa} ker(s_\alpha)$, it follows that $ker(s)$ is also contained in a κ-block of A_α. Thus, $s \in W^{\mathscr{I}}$. Now suppose that $W^{\mathscr{I}} \models \forall x \exists y \varphi(x, y, u)$, where $u \in W^{\mathscr{I}}$. Then for every $s \in (W^{\mathscr{I}})^{<\kappa}$ there is some $y \in W^{\mathscr{I}}$ such that $\varphi^{W^{\mathscr{I}}}(s, y, u)$. By the DC_κ-scheme in U, there is an $f : \kappa \to W^{\mathscr{I}}$ with $\varphi^{W^{\mathscr{I}}}(f \restriction \alpha, f_\alpha, u)$ for every $\alpha < \kappa$. Since $f \in W^{\mathscr{I}}$, it follows that $W^{\mathscr{I}} \models \mathrm{DC}_\kappa$-scheme. □

Next we show that Reflection Principle holds in $W^{\mathscr{I}}$. We shall utilize the following observation made in [6, page 397] (though in a different context).

Theorem 9. $\mathrm{ZFU_R} \vdash$ Collection $\land$ DC-scheme $\to$ RP. □

Thus, to show $W^{\mathscr{I}} \models$ Reflection Principle it suffices to check that $W^{\mathscr{I}}$ satisfies Collection. A permutation σ of $\mathscr{A}$ in U is said to be *row-swapping* if for every $\alpha < \kappa^+$, $\sigma A_\alpha = A_\beta$ for some $\beta < \kappa^+$ (consequently, for every α, $A_\alpha = \sigma A_\beta$ from some β).

Lemma 10. Every row-swappoing σ in U is an automorphism of $W^{\mathscr{I}}$.

Proof. Let σ be a row-swappoing permutation of $\mathscr{A}$. It suffices to show that $\mathcal{G}_{\mathscr{A}}$, I, and $\mathscr{I}$ are all fixed by σ. If $\pi \in \mathcal{G}_{\mathscr{A}}$, then for every A_α, since $A_\alpha = \sigma A_\beta$ for some β and $\pi A_\beta = A_\beta$, by automorphism it follows that $(\sigma\pi)(\sigma A_\beta) = \sigma A_\beta$; so $(\sigma\pi)A_\alpha = A_\alpha$ and hence $\sigma\pi \in \mathcal{G}_{\mathscr{A}}$ (note here $\sigma\pi$ is not $\sigma \circ \pi$ but $\{\langle \sigma a, \sigma(\pi a)\rangle : a \in \mathscr{A}\rangle\}$). This shows that $\sigma\mathcal{G}_{\mathscr{A}} = \mathcal{G}_{\mathscr{A}}$. If $E \in I$, then for every A_α, since $A_\alpha = \sigma A_\beta$ for some β and $E \cap A_\beta$ has size $< \kappa^+$, $\sigma E \cap A_\alpha$ has size $< \kappa^+$ and hence $\sigma E \in I$. Therefore, $\sigma I = I$. Similarly, if B is contained in κ-many A_α, then so is σB. Hence, $\sigma\mathscr{I} = \mathscr{I}$ and the lemma is proved. □

Lemma 11. $W^{\mathscr{I}} \models$ RP.

Proof. We show that $W^{\mathscr{I}} \models$ Collection. Suppose that $W^{\mathscr{I}} \models \forall x \in w \exists y \varphi(x, y, u)$ for some $w, u \in W^{\mathscr{I}}$. Let $A = \bigcup_{\alpha<\kappa} A_\alpha$ be a κ-block containing $ker(w) \cup ker(u)$ and $B = \bigcup_{\alpha<\kappa} B_\alpha$ be another κ-block that is disjoint from A. It is enough to show that $W^{\mathscr{I}} \models \forall x \in w \exists y \in V(A \cup B)\varphi(x, y, u)$ because then a sufficiently tall $V_\alpha(A \cup B)$ will be a desired collection set.

Consider any $x \in w$ and $y \in W^{\mathscr{I}}$ such that $W^{\mathscr{I}} \models \varphi(x, y, u)$. Let $C = \bigcup_{\alpha<\kappa} C_\alpha$ be another κ-block containing $ker(y) - A$.

Claim 11.1. In U there is a row-swapping σ such that $\sigma(B \cup C) = B$.

Proof of the Claim. Fix a κ-block $D = \bigcup_{\alpha<\kappa} D_\alpha$ disjoint from $A \cup B \cup C$. Split B into two pair-wise disjoint κ-blocks $B^1 = \bigcup_{\alpha<\kappa} B^1_\alpha$ and $B^2 = \bigcup_{\alpha<\kappa} B^2_\alpha$, where each B^1_α (and B^2_α) is some row A_α. Then split D into two pair-wise disjoint κ-blocks $D^1 = \bigcup_{\alpha<\kappa} D^1_\alpha$ and $D^2 = \bigcup_{\alpha<\kappa} D^2_\alpha$ in the same way. In U we can define a row-swapping permutation σ of $\mathscr{A}$ as follows. For each $\alpha < \kappa^+$, let $\sigma B_\alpha = B^1_\alpha$, $\sigma C_\alpha = B^2_\alpha$, $\sigma D^1_\alpha = C_\alpha$ and $\sigma D^2_\alpha = D_\alpha$. It is clear that $\sigma(B \cup C) = B$. ■

By Lemma 10 σ is an automorphism of $W^{\mathscr{I}}$ fixing x and u, so $W^{\mathscr{I}} \models \varphi(x, \sigma y, u) \wedge \sigma y \in V(A \cup B)$, which proves the lemma. □

Finally, we turn to the failure of Hume's Principle in $W^{\mathscr{I}}$. Suppose *for reductio* that $\varphi(x, \#x, u)$ defines a cardinality-assignment map with some parameter $u \in W^{\mathscr{I}}$. It follows that there must be two A_α and A_β that are disjoint from $ker(u)$.

Claim 11.2. $W^{\mathscr{I}} \models A_\alpha \nsim A_\beta$.

Proof of the Claim. Suppose *for reductio* that f is an injection from A_α to A_β in $W^{\mathscr{I}}$. Let $E \in I$ be a support of f. Then there are two urelements $a, b \in A_\alpha - E$. Let $\pi \in fix(E)$ swap only a and b. Then $f(b) = \pi f(a) = f(a)$, contradicting the assumption that f is an injection. ■

Claim 11.3. $ker(\#A_\alpha) \subseteq ker(u)$.

Proof of the Claim. Suppose not. Fix some $a \in ker(\#A_\alpha) - ker(u)$ and some $b \notin ker(u) \cup ker(\#A_\alpha)$. In $W^{\mathscr{I}}$ let π be an automorphism that only swaps a and b. Since π fixes u and $\pi A_\alpha \sim A_\alpha$, we have $\varphi(\pi A_\alpha, \pi\#A_\alpha, u)$ so $\pi\#A_\alpha = \#\pi A_\alpha = \#A_\alpha$. But $\pi\#A_\alpha \neq \#A_\alpha$—contradiction. ■

By Lemma 10, in U there is an automorphism σ of $W^{\mathscr{I}}$ that swaps only A_α and A_β. As σ point-wise fixes $ker(u)$ and hence $ker(\#A_\alpha)$, it follows that $W^{\mathscr{I}} \models \varphi(A_\beta, \#A_\alpha, u)$ so $\#A_\beta = \#A_\alpha$. Hence, $W^{\mathscr{I}} \models A_\alpha \sim A_\beta$, contradicting Claim 11.2. This completes the proof of the theorem. □

4.2. **Second-Order Hume's Principle.** A map $X \mapsto \#X$ from classes to sets fulfills Hume's Principle *simpliciter* if it maps each class X to a first-order object $\#X$ such that two classes X and Y are equinumerous just in case $\#X = \#Y$. In class theory, von Neumann's Axiom of Limitation of Size states that all proper classes are equinumerous. Limitation of Size implies the Axiom of Global Well-Ordering, which asserts the existence of a well-ordering of the universe U, and the full Hume's Principle since we can map each set to its well-ordered cardinal and all proper classes to a fixed object which is not an ordinal, say, $\{\{\emptyset\}\}$. A standard argument shows that Global Well-Ordering is equivalent to Limitation of Size when the urelements form a set ([17, Proposition 98]). We prove in the next theorem that Global Well-Ordering alone does not suffice for Hume's Principle.

Theorem 12. Assume the consistency of ZFC + an inaccessible cardinal. There is a model of KM class theory with urelements + Global Well-Ordering which have no definable map that fulfills Hume's Principle.

Proof. Let V be a model of ZFC + an inaccessible cardinal κ. We first construct a model U of $\mathrm{ZFCU_R}$ + Plenitude by treating copies of ordinals as urelements. In particular, we define $V[\![Ord]\!]$ in V by recursion as follows.

$$V[\![Ord]\!] = (\{0\} \times Ord) \cup \{\bar{x} \in V : \exists x(\bar{x} = \langle 1, x\rangle \wedge x \subseteq V[\![Ord]\!])\}.$$

For every $\bar{x}, \bar{y} \in V[\![Ord]\!]$,

$\bar{x} \mathrel{\bar{\in}} \bar{y}$ if and only if $\exists y(\bar{y} = \langle 1, y\rangle \wedge \bar{x} \in y)$;

$\bar{\mathscr{A}}(\bar{x})$ if and only if $\bar{x} \in \{0\} \times Ord$.

Let U denote the model $\langle V[\![Ord]\!],\ \bar{\mathscr{A}},\ \bar{\in}\rangle$ for the language of urelement set theory. By [17, Theorem 7 and 9], $U \models \mathrm{ZFCU_R}$ + Plenitude. Moreover, the class of pure sets in U is isomorphic to V ([17, Lemma 8]) so U also contains (a copy of) κ as an inaccessible cardinal.

In U, let $\lambda = \aleph_{\kappa^+}$ and A be a set of urelements of size λ. Define

$$U_\kappa(A) = \bigcup_{B \in P_\kappa(A)} V_\kappa(B),$$

where $P_\kappa(A)$ is the set of subsets of A with size $< \kappa$. By [16, Lemma 4.2], $U_\kappa(A)$, equipped with all of its subsets, is a model of KM with urelements + Global Well-Ordering. Now suppose *for reductio* that $\varphi(X, \#X)$ defines a cardinality-assignment function in $U_\kappa(A)$.

Claim 12.1. For every class X of $U_\kappa(A)$, $\#X$ is a pure set.

Proof of the Claim. Suppose $\#X$ is not pure for some class X of $U_\kappa(A)$. Then in $U_\kappa(A)$ there will be an automorphism π swapping some $a \in ker(\#X)$ with some $b \notin ker(\#X)$, and so $\pi\#X \neq \#X$. Since $\pi X \sim X$, it follows that $\#X = \#\pi X = \pi\#X$, yielding a contradiction. ■

Since there are at least κ^+-many cardinalities for the proper classes of $U_\kappa(A)$, φ thus defines an injection from κ^+ into $V^{U_\kappa(A)}$, namley, V_κ. This is impossible as V_κ has size κ. □

This argument, in fact, shows that in $U_\kappa(A)$ there is no definable cardinality-assignment map which only uses first-order objects in $U_\kappa(A)$ as parameters. Suppose $\varphi(X, \#X, u)$ defines a cardinality-assignment map, where $u \in U_\kappa(A)$. Then the same argument would show that $ker(\#X) \subseteq \text{ker}(u)$ for every class X. This means that we could inject κ^+-many cardinalities into $V_\kappa(ker(u))$, but $V_\kappa(ker(u))$ only has size κ as $|ker(u)| < \kappa$—contradiction.

4.3. **Philosophical remarks.** We end with an discussion of the philosophical implications these results might have on Hume's Principle. Of course, to do so we must adopt the so-called *external perspective* (Shapiro [14]), which is to investigate abstraction principles through other mathematical theories with presumably stronger interpretive power. As we noted in the introduction, one necessary condition for Hume's Principle to be regarded as a fundamental principle of cardinality is its *universality*, i.e., the opening quantifiers in Hume's Principle must range over concepts of any kind. Consequently, models of urelement set theory are precisely where Hume's Principle should be tested from the external perspective. We have seen models with urelements where Hume's Principle fails, but whether this constitutes evidence against the universality of Hume's Principle depends on the naturalness of these models. Therefore, the question remains: are these counter-models for Hume's Principle natural? Two features are shared by these models: the urelements form a proper class, and a certain form of Axiom of Choice fails. The first feature are certainly not be considered as pathological, as Hume's Principle should not be contingent upon how many urelements there are. The second feature in these models, however, is worth remarking on.

Consider first the model $W^{\mathscr{I}}$ of Theorem 7, where Hume's Principle fails for sets. Although $W^{\mathscr{I}}$ does not satisfy AC, I argue that this does not make it a pathological model. The justification for AC as a mathematical axiom often stems from its ability to make mathematical objects well-behaved and to produce desirable consequences, despite its paradoxical consequences. However, in urelement set theory, AC seems to assert more than necessary, as it excludes any non-well-orderable sets even if these sets do not exist within the mathematical universe V. Furthermore, it is clear that AC holds for the pure sets in $W^{\mathscr{I}}$ since these pure sets remain unchanged throughout the construction. Consequently, not only does $W^{\mathscr{I}}$ satisfy a strong fragment of AC, but the non-well-orderable sets in $W^{\mathscr{I}}$ also originate from outside its mathematical universe V. This makes $W^{\mathscr{I}}$ a mathematically natural model, even for those who advocate for AC as a well-justified axiom in pure set theory.

Next, consider the model $U_\kappa(A)$ of Theorem 12, where the full Hume's Principle fails. The fact that Global Well-Ordering holds in $U_\kappa(A)$ makes it an extremely natural model of urelement class theory, which provides strong evidence against the unversality of Hume's Principle. This is because Global Well-Ordering has highly desirable consequences, including the Reflection Principle, the Collection Principle, and the DC_κ-scheme for every κ, which are provable in pure class theory but not in urelement class theory when we only assume AC for sets ([17, Proposition 101]) In comparison with Global Well-Ordering, Limitation of Size, although it implies Hume's Principle, is rather *ad hoc* in this context. For one thing, assuming that all proper classes are equinumerous simply lacks independent motivations. For another, it is shown in Hamkins & Yao [8] that Limitation of Size places a significant constraint on the consistency strength of Bernays' Second-order Reflection Principle, which is an extremely natural large cardinal axiom. So in urelement class theory, it is Global Well-Ordering that should be considered as the standard second-order generalization of AC.

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