Reflective Mereology

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Abstract

I propose a new theory of mereology based on a mereological reflection principle. Reflective mereology has natural fusion principles but also refutes certain principles of classical mereology such as Universal Fusion and Fusion Uniqueness. Moreover, reflective mereology avoids Uzquiano's cardinality problem–the problem that classical mereology tends to clash with set theory when they both quantify over everything. In particular, assuming large cardinals, I construct a model of reflective mereology and second-order ZFCU with Limitation of Size. In the model, classical mereology holds when the quantifiers are restricted to the urelements.

1 Introduction

In [28] and [29], Uzquiano observes that atomistic classical mereology is in tension with set theory when both theories quantify over absolutely everything. It is thus interesting to see if there can be a well-motivated mereology that sits well with set theory. In this paper, I propose a new mereology based on a mereological reflection principle. This new mereology is shown to have various interesting consequences and sit well with set theory.

The rest of the paper unfolds as follows. In Section 2, I lay out the formal setting and present Uzquiano's cardinality problem. In particular, I show that atomistic classical mereology is inconsistent with set theory with the principle of Limitation of Size when they share the same domain of quantification. In Section 3, I first motivate a mereological reflection principle, based on which I introduce a new theory of parthood–reflective mereology. Reflective mereology is then shown to disprove two core principles in classical mereology: Unrestricted Fusion and Weak Supplementation. Section 4 studies the interaction between reflective mereology and set theory. I show that a natural strengthening of reflective mereology implies a mereological analog of the axiom of replacement; with Limitation of Size, reflective mereology implies that things have a fusion just in case they form a set. Furthermore, many axioms of set theory and the existence of certain large cardinals can also be derived from reflective mereology plus Limitation of Size. In Section 5, I show that reflective mereology is consistent with set theory assuming the consistency of second-order set-theoretic reflection. This is done by interpreting proper parthood in the set-theoretic universe with urelements, which produces a natural model of both reflective mereology and set theory. Moreover, in this model, classical mereology holds when the quantifiers are restricted to the urelements. In Section 6, I discuss several philosophical issues related to the formal results proved. In particular, I argue that reflective mereology is a well-motivated theory of parthood.

2 Preliminaries and the Cardinality Problem

2.1 The formal language

We shall work in the language of plural quantification denoted by $\mathscr{L}^{\propto}$. In addition to the logical symbols of first-order language, $\mathscr{L}^{\propto}$ contains plural variables "$xx, yy, zz, \ldots$" and a predicate "$\propto$". "$x \propto yy$" stands for "x is among yy". Every instance of the following comprehension scheme is always an axiom in any extension of $\mathscr{L}^{\propto}$:

(*Plural Comprehension*) $\exists yy \forall z(z \propto yy \leftrightarrow \psi(z))$,

where $\psi(z)$ is a formula that does not contain any free occurrence of yy. Since the empty plurality is allowed to exist (a similar treatment can be found in Burgess [3]), our treatment of plural quantification is completely analogous to monadic second-order logic [2]. The talk of pluralities is preferred because I wish to avoid talking about classes in the meta-language. We shall not specify a deductive system for $\mathscr{L}^{\propto}$ since any standard deductive system for second-order logic will suffice (for example, see Button and Walsh [31], p. 34).

The plural language of mereology $\mathscr{L}^{\propto}_{<}$ extends $\mathscr{L}^{\propto}$ by adding a binary first-order predicate "$<$" which stands for "is a proper part of" (for recent studies on plural logic and mereology, see [9]) and [8]).

Definition 2.1. $xx \neq \emptyset$ (xx are non-empty) $=_{df} \exists y(y \propto xx)$
$xx \propto\propto yy$ (xx are among yy) $=_{df} \forall z \propto xx(z \propto yy)$ [1]
$xx \ll y$ (xx are proper parts of y) $=_{df} \forall z \propto xx(z < y)$
$x \leqslant y$ (x is a part of y) $=_{df} x < y \vee x = y$
$xx \leqslant\leqslant y =_{df} \forall x \propto xx(x \leqslant y)$
$Atom(x)$ (x is a mereological atom) $=_{df} \forall y(y \nless x)$[2]
$x \circ y$ (x overlaps y) $=_{df} \exists z(z \leqslant x \wedge z \leqslant y)$
$Fu(x, yy)$ (x is a fusion of yy) $=_{df} \forall y \propto yy(y \leqslant x) \wedge \forall z \leqslant x \exists w \propto yy(w \circ z)$
yy fuse $=_{df} \exists x Fu(x, yy)$

It is only a matter of convenience that proper parthood is chosen to be primitive in $\mathscr{L}^{\propto}_{<}$: in the new mereology that will be introduced later, it is proper parthood, rather than parthood, that is used in the formulation of a key axiom. Alternatively, one can still use the parthood symbol "$\leq$" as the primitive symbol, then define "$x < y$" as "$x \leq y \wedge x \neq y$". Our definition of fusion is standard as in Tarski [25] [26] and Lewis [17]. It is immediate from the definition that everything fuses itself [3] as well as the plurality of its proper parts.

$\mathscr{L}^{\propto}_{\in,\mathrm{Ur}}$ is the language of set theory with urelements which extends $\mathscr{L}^{\propto}$ by adding a unary first-order predicate Ur and the membership relation $\in$. Ur stands for "is a urelement". The following definitions are completely standard, which will be used in Section 4.

Definition 2.2. $Set(x) =_{df} \neg\mathrm{Ur}(x)$
$x \subseteq y =_{df} Set(x) \wedge \forall z \in x(z \in y)$
xx form a set $=_{df} \exists x(Set(x) \wedge \forall y(y \in x \leftrightarrow y \propto xx))$

[1] To ease the notation, most of time I will write $\forall\varphi(x)\psi$ as shorthand for $\forall x(\varphi(x) \to \psi)$ and write $\exists\varphi(x)\psi$ for $\exists x(\varphi(x) \wedge \psi)$. Plural quantification is treated similarly.

[2] In this paper, "atoms" should always be understood as mereological atoms rather than urelements.

[3] More pedantically, every x is a fusion of the "improper" plurality that only consists of x. I shall omit this subtlety from now on.

$xx \in\in y =_{df} \forall x \propto xx(x \in y)$
$x \propto xx \cap y =_{df} x \propto xx \wedge x \in y$
$z = x \cup \{x\} =_{df} x \in z \wedge \forall v \in x(v \in z) \wedge \forall v \in z(v = x \vee v \in x)$
$z = \{x,y\} =_{df} x \in z \wedge y \in z \wedge \forall w \in z(w = x \vee w = y)$
$z = \langle x,y \rangle =_{df} \exists w \in z(w = \{x\}) \wedge \exists v \in z(w = \{x,y\}) \wedge \forall v \in z(v = \{x\} \vee v = \{x,y\})$
$\langle x,y \rangle \propto ff =_{df} \exists z \propto ff(z = \langle x,y \rangle)$
$ff : xx \longrightarrow yy =_{df} \forall z \propto ff \exists x \propto xx \exists y \propto yy(z = \langle x,y \rangle) \wedge \forall x \propto xx \exists! y(\langle x,y \rangle \propto ff)$
$ff : xx \xrightarrow{\text{onto}} yy =_{df} ff : xx \rightarrow yy \wedge \forall y \propto yy \exists x(\langle x,y \rangle \propto ff)$
$yy \preceq xx =_{df} \exists ff(ff : xx \xrightarrow{\text{onto}} yy)$

Definition 2.3. The theory $\mathbf{ZFCU}_2$ consists of the following axioms (it is understood that the formulas are taken to be their universal closure).

1. (Ur-Def) $\mathrm{Ur}(x) \rightarrow \forall y(y \notin x)$
2. (Ext) $(Set(x) \wedge Set(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \rightarrow x = y$
3. (Foundation) $Set(x) \wedge \exists y(y \in x) \rightarrow \exists z \in x \forall v \in x(v \notin z)$
4. (Pairing) $\exists z(Set(z) \wedge z = \{x,y\})$
5. (Union) $Set(x) \rightarrow \exists z[Set(z) \wedge \forall w(w \in z \leftrightarrow \exists y \in x(w \in y))]$
6. ($\in$-Separation) $xx \propto\propto yy \wedge yy$ form a set $\rightarrow xx$ form a set
7. ($\in$-Replacement) $xx \preceq yy \wedge yy$ form a set $\rightarrow xx$ form a set
8. (Infinity) $\exists x[\exists y \in x(Set(y) \wedge \forall z(z \notin y)) \wedge \forall w \in x(w \cup \{w\} \in x)]$
9. (Powerset) $Set(x) \rightarrow \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$
10. (The Axiom of Choice) Every set can be well-ordered.

Definition 2.4. The language $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$ denotes $\mathscr{L}^{\propto}_{<} \cup \mathscr{L}^{\propto}_{\in,Ur}$

2.2 Uzquiano's cardinality problem

It is observed in Uzquiano [28] and [29] that plural classical mereology is in tension with plural ZFC set theory when their quantifiers are both taken to be absolutely general. In this section, I give a new presentation of Uzquiano's cardinality problem.

We shall work in the hybrid language $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$, where mereological and set-theoretic principles share the same domain of quantification .

Definition 2.5. *Classical Mereology* (**CM**) consists of the following four axioms.

(Transitivity) $x < y \wedge y < z \rightarrow x < z$

(Asymmetry) $x < y \rightarrow y \not< x$

(Weak-Supplementation) $x < y \rightarrow \exists z < y \neg(z \circ x)$

(Unrestricted Fusion) $xx \neq \emptyset \rightarrow xx$ fuse[4]

CM is an attractive theory of parthood adopted by Tarski [25] [26] and Lewis [17] [16]. Transitivity and Asymmetry will be taken for granted in this paper, as they are intuitive principles governing proper parthood. Weak Supplementation is a plausible *decomposition principle*: if x is a proper part of y, then y must have something more, namely, a leftover z disjoint from x. This picture coheres with our conception of ordinary objects, and some (e.g., Simons [23], p. 116 and Varzi [30] p. 110) even hold Weak Supplementation as an analytic truth. The philosophical motivations for Unrestricted Fusion will be discussed shortly.

It follows from **CM** that fusions are unique (Hovda [13, pp. 66-67]), namely:

(Fusion Uniqueness) $Fu(y,xx) \wedge Fu(z,xx) \rightarrow y = z$

Let uu denote the plurality of everything (i.e., $x \propto uu$ iff $x = x$). Uzquiano's cardinality arises when I consider two further principles.

(LS) xx form a set $\leftrightarrow uu \not\preceq xx$.

[4]Although Asymmetry follows from Weak-Supplementation and Transitivity, it is useful to separate them for reasons that will be clear.

(Atomicity) $\exists y \leqslant x Atom(y)$

LS is known as von Neumann's *Limitation of Size*, according to which a plurality of things *xx* form a set just in case there is no surjective map from *xx* to everything there is. In other words, things will form a set as long as they are not "too many". Limitation of Size seems to be a natural conception of set and provides a unified justification for important axioms of $\mathbf{ZFCU}_2$ such as $\in$-Replacement and the Axiom of Choice. Atomicity describes an intuitive mereological structure since it is equivalent to the claim that everything is a fusion of some mereological atoms (as we will see, the full strength of Atomicity is not needed for the paradox to arise).

Now, Uzquiano's cardinality problem is simply that the theory $\mathbf{ZFCU}_2$ + LS + **CM** + Atomicity is inconsistent. The proof can be sketched as follows. By **CM**, there must be more fusions than the atoms, so the atoms form a set by LS because they are not too many. But everything is a fusion of some atoms by Atomicity, and by Fusion Uniqueness a set of atoms can only produce a set of things. Therefore, the atoms must not form a set—-a contradiction. To make this precise, let us start with a more general definition.

Definition 2.6. *xx* is a $<$-antichain $=_{df} \forall v, w \propto xx(v \neq w \to \neg(v \circ w))$

We show that with LS, $\mathbf{ZFCU}_2$ + **CM** proves that every $<$-antichain must form a set. The proof is by diagonalization.

Lemma 2.1. $\mathbf{ZFCU}_2$ + **CM** + LS $\vdash \forall xx(xx$ is a $<$-antichain $\to xx$ form a set)

Proof. Let *aa* be a $<$-antichain. We may assume that there are at least two things among *aa*; otherwise *aa* form a set by Pairing (or, by the existence of the empty set). Suppose for *reductio* that there is ff such that $ff : aa \xrightarrow{\text{onto}} uu$, where *uu* is the plurality of everything. Now define *dd* as follows.

$$x \propto dd \leftrightarrow x \propto aa \wedge x \not< ff(x)$$

Note that *dd* is non-empty. To see this, consider any $a \propto aa$. Since ff is onto, there is some $b \propto aa$ such that $ff(b) = a$. If $b = a$, then $b \not< a$ by Asymmetry; otherwise, b does not overlap a and hence $b \not< a$. Thus, $b \not< ff(b)$ and so $b \propto dd$. In fact, there are at least two things among dd. Consider two different $a_1, a_2 \propto aa$. Then there will be $b_1, b_2 \propto aa$ such that $ff(b_1) = a_1$ and $ff(b_2) = a_2$; so $b_1 \neq b_2$ and $b_1, b_2 \propto dd$.

By Unrestricted Fusion, some z fuses dd. Then there is some $c \propto aa$ such that $ff(c) = z$. If $c < z$, c must overlap some d among dd; $d \propto aa$, so $c = d$ and hence $c \nless ff(c) = z$, which is a contradiction. So $c \nless z$. Then $c \propto dd$ so $c \leqslant z$. But $c \neq z$. Otherwise, since dd contain at least two things, there will be some $e \propto dd$ such that $e \neq c \wedge e \leqslant c$, which is impossible because both e and c are among the antichain aa. This means that $c < z$, which is a contradiction again.

Therefore, there is no ff such that $ff : aa \xrightarrow{\text{onto}} uu$, i.e., $uu \npreceq aa$. By LS, aa form a set. □

We then show that under $\mathbf{ZFCU}_2$ + **CM** + Atomicity, the mereological atoms cannot form a set.

Lemma 2.2. $\mathbf{ZFCU}_2$ + **CM** + Atomicity $\vdash \neg\exists x(Set(x) \wedge \forall y(Atom(y) \to y \in x))$

Proof. Let aa be the plurality of all atoms. Suppose for *reductio* that aa form a set, A. Then the power set $P(A)$ exists. Define $ff : P(A) \setminus \{0\} \to uu$ as follows. For every non-empty $x \subseteq A$, let $ff(x)$ be the fusion of the members of x. ff is well-defined by Unrestricted Fusion and Fusion Uniqueness. We claim that ff is also onto the plurality of everything uu. For every z, $Fu(z, yy)$ for some atoms yy by Atomicity; since A is the set of all atoms, by $\in$-Separation some set $y \subseteq A$ will be the set of yy and so $ff(y) = z$. By $\in$-Replacement, it follows that uu form a set, which is a contradiction by Russell's paradox. □

Theorem 2.3. $\mathbf{ZFCU}_2$ + LS + **CM** + Atomicity is inconsistent.

Proof. Let aa be the plurality of all atoms, which is clearly a $<$-antichain. By Lemma 2.1, aa form a set, but this contradicts Lemma 2.2. [5] □

[5] Our presentation is different from Uzquiano's in two ways. First, Uzquiano's argument is based on the following principle, which seems to be weaker than LS.

(Maximality) There is an injective map from the universe to the pure sets.

LS implies Maximality. For, LS implies that if xx and yy both fail to form a set, then there is a bijective map between them; and neither the pure sets nor the universe could form a set. However, it is unclear

The argument above in fact shows that under $\mathbf{ZFCU}_2$ + LS + **CM**, there can be at most a set of atoms and so there must be proper-class-many atomless gunks. Appealing to Atomicity is thus an overkill: as long as there are not enough atomless gunks, the theory $\mathbf{ZFCU}_2$ + LS + **CM** is inconsistent. Uzquiano considers different attempts to resolve this unwelcome situation in [28] and [29] but does not find any of them satisfactory. In the rest of this paper, I aim to solve Uzquiano's cardinality problem by proposing *a new mereology* that can coexist with set theory in harmony.

Before turning to this new mereology, it is worth considering how axioms of mereology can be justified. While to fully address this question will take us too far afield, it may be useful to reflect on **CM**, which is often seen as a paradigmatic theory of parthood. Firstly, it should be a virtue of a theory of parthood if it is compatible with the existence of ordinary objects. And since things always have a fusion according to **CM**, ordinary composite objects are preserved by **CM** in our ontology. Secondly, an axiom of mereology, if not completely intuitive, should be motivated by independent philosophical considerations, otherwise the axiom might be seen as *ad hoc*. Two philosophical considerations motivate Unrestricted Fusion. One is that there should be no ontological arbitrariness regarding mereological fusion (Lewis [17] [16]), and this is perfectly respected by Unrestricted Fusion since it asserts that fusion always happens. The other consideration is often known as *Composition as Identity*—the doctrine that the mereological fusion of some parts is nothing "over and above" the parts, i.e., once we are committed to the parts, we are immediately committed to their mereological fusion (see Sider [22] for more on this). While there is no space to fully evaluate these two philosophical views, it is safe to say that they certainly provide enough motivations for **CM**. That said, there are at least two desiderata for a well-motivated mereology: (i) it should have fusion principles that are compatible with ordinary objects; (ii) its axioms should be motivated by independent philosophical consid-

whether $\mathbf{ZFCU}_2$ + **CM** + Atomicity + Maximality is inconsistent (the theory will indeed be inconsistent if we assume that Limitation of Size holds for the pure sets, but then the full Limitation of Size becomes equivalent to Maximality). Yet this point does not concern us: reflective mereology will be proved to be consistent with LS and hence with Maximality.

Second, it is claimed in [28] and [29] that with Maximality, the size of the universe is *inaccessible* in second-order terms. Again, without full Limitation of Size, this is not true (see this question on MathStackExchange: `https://math.stackexchange.com/questions/4174937/show-that-v-is-a-strong-limit-in-second-order-t`).

erations. By Theorem 2.3, we know that any mereology that is consistent with $\mathbf{ZFCU}_2$ + LS + Atomicity must differ with **CM** to some extent. So if a well-motivated mereology that avoids the cardinality problem is to be found, it will presumably have some weaker fusion principles and be motivated by some different philosophical theses. But ideally, this new mereology will inherit some merits of **CM** and at the same time provide reasons for why **CM** fails when it is formulated with absolute generality.

3 Reflective Mereology

3.1 Reflection principles in set theory

The view that the universe of sets is "absolutely indescrible", which can be traced back to Cantor[4], asserts that any true statement about the universe of sets is already true in some initial fragment of the universe. In contemporary set theory, this indescribability conception of set is cashed out by the set-theoretic *reflection principles*, according to which any true set-theoretic statement is already true in some *transitive* set (where a set is transitive if every member of it is a subset of it). What is it to say that a statement is *true in a set*? The standard definition is the following.

Definition 3.1. For any s and any formula ψ in the language $\mathscr{L}_{\in}^{\propto}$, $\psi^{\in s}$ is defined inductively as follows.

If ψ is an atomic sentence other than $x \propto yy$, $\psi^{\in s}$ is ψ;

if ψ is $x \propto yy$, $\psi^{\in s}$ is $x \propto yy \cap s$;

if ψ is $\varphi \vee \chi$, $\psi^{\in s}$ is $\varphi^{\in s} \vee \chi^{\in s}$;

if ψ is $\neg\varphi$, $\psi^{\in s}$ is $\neg\varphi^{\in s}$;

if ψ is $\exists x\varphi$, $\psi^{\in s}$ is $\exists x \in s\varphi^{\in s}$;

if ψ is $\exists xx\varphi$, $\psi^{\in s}$ is $\exists xx \in\in s\varphi^{\in s}$.

Intuitively, $\psi^{\in s}$ is what ψ would assert if we were to take the members of s to be all there is, and to say that ψ *is true in* s is to say that $\psi^{\in s}$ holds. The second-order reflection principle, due to Bernays [1], is the following scheme in $\mathbf{ZFCU}_2$.

(RP$_2$) $\psi \to \exists s(s \text{ is a transitive set} \wedge \psi^{\in s})$.[6]

According to RP$_2$, no statement can pin down the universe of sets since whatever holds in the universe already holds in a transitive set. While a thorough discussion of reflection principles is beyond the scope of this paper, there are mainly two motivations for RP$_2$. First, just as Limitation of Size, RP$_2$ provides a unified justification for almost all axioms of second-order ZF: Bernays [1] shows that RP$_2$, $\in$-Separation, Ext and Foundation jointly imply all the axioms of second-order ZF and the existence of several large cardinals. Second, it is common for set theorists to hold that the universe of sets is *maximal* in the sense that it contains as many mathematical objects as possible. RP$_2$, as a way of turning this philosophical view into a precise mathematical statement, is believed by many (Gödel[10], Maddy [18] [19], Tait [24], Koellner [15]) to be *intrinsically justified*, namely, it is justified on the basis of the conception of set. Gödel[10], for example, believes that all large cardinal axioms should follow from certain form of reflection principles.[7] It is reasonable, therefore, to adopt the indescribability conception of set articulated by RP$_2$.

3.2 Reflective mereology

The indescribability of the set-theoretic universe naturally gives rise to a more general metaphysical thesis: the universe of *everything*, which includes both sets and non-sets, is also indescribable. In particular, the universe cannot be described in any *mereological*

[6]This particular version is formulated by Burgess[3] in the language of set theory with plural quantification. More rigorously, RP$_2$ is the following axiom scheme

$$\forall x_0,\dots,x_n \forall xx_0,\dots,xx_m(\psi(x_0,\dots,x_n,xx_0,\dots,xx_m) \to \exists t(t \text{ is a transitive set} \wedge \psi^{\in t}(x_0,\dots,x_n,xx_0,\dots,xx_m)),$$

where ψ is a formula in $\mathscr{L}_{\in}^{\propto}$ whose free variables are among $x_0,\dots,x_n,xx_0,\dots,xx_m$.

[7]In fact, RP$_2$ is quite limited in its power, as its consistency strength is bounded by some small large cardinal at the lower end of the large-cardinal hierarchy (e.g., an ω-Erdös cardinal suffices, see Kanamori [14] Excercise 9.18. However, we note that RP$_2$ can be quite strong if we expand the language (see Robert[20]) or assume the existence of a lot of urelements (see Hamkins and Yao[12]).

terms. This gives rise to the Mereological Reflection Principle: any true statement is already true in some *parthood initial segment* of the universe. But note that since everything is a $<$-initial segment given Transitivity, the Mereological Reflection Principle even has a more concise formulation:

> Any true statement is already true in something.

Of course, here "true in" needs to be spelled out in the language of mereology. Fortunately, there is no difficulty of making this sense of "true in" precise by considering a form of $<$-*relativization*. To begin with, we define the mereological relativization of plural terms. Plural Comprehension allows us to talk about arbitrary pluralities, e.g., the plurality of everything. Now suppose that we live in an object o and thus only have access to its proper parts; then the meaning of a plural term should change with respect to o. For example, "everything" from the perspective of people in o should be all the proper parts of o. This motivates the following definition.

Definition 3.2. For any yy and any x, yy^x (the $<$-relativization of yy to x) denotes the proper parts of x among yy, i.e., $z \propto yy^x$ if and only if $z \propto yy \wedge z < x$.[8]

Furthermore, given any statement ψ and object x, we can simply restrict all the quantifiers and plural terms in ψ to the proper parts of x. The resulting statement, $\psi^{<x}$, will be what ψ would mean if the proper parts of x were all there is. So "ψ is true in x" simply means that $\psi^{<x}$ holds. The formal definition goes as follows.

Definition 3.3. For any t and any statement ψ is in $\mathscr{L}^{\propto}_{<}$, $\psi^{<t}$ is defined inductively as follows.

If ψ is an atomic formula other than $x \propto yy$, $\psi^{<t} = \psi$;

if ψ is $x \propto yy$, $\psi^{<t} = x \propto yy^t$;

if ψ is $\varphi \vee \chi$, $\psi^{<t} = \varphi^{<t} \vee \chi^{<t}$;

if ψ is $\neg\varphi$, $\psi^{<t} = \neg\varphi^{<t}$;

[8] Note that yy^x can be empty, which is allowed in our plural logic.

if ψ is $\exists x\varphi$, $\psi^{<t} = \exists x < t\varphi^{<t}$;

if ψ is $\exists xx\varphi$, $\psi^{<t} = \exists xx \ll t\varphi^{<t}$.[9]

Now we can give a precise formulation of the Mereological Reflection Principle.

Definition 3.4. The *Mereological Reflection Principle* (MRP) is the following scheme in $\mathscr{L}^{\propto}_{<}$.

(MRP) $\psi \to \exists t\psi^{<t}$.[10]

A quick remark: it is important that we relativize to the *proper* parts of t. If $<$ is replaced with $\leqslant$, MRP will become a trivial consequence of **CM**, as ψ always implies $\psi^{\leqslant u}$, where u is the fusion of everything.

Is MRP a well-motiaved axiom of mereology? It seems that MRP indeed satisfies the first desideratum proposed at the end of Section 2, namely, it is motivated by some independent philosophical consideration. In particular, the metaphysical thesis underlying MRP seems to be clear, i.e., the whole universe, which includes absolutely everything, is indescribable. Moreover, this metaphysical picture is a natural generalization of the *Cantorian* conception of set which is commonly accepted by set theorists. Although whether the universe of everything *should be* similar to the universe of sets in terms of indescribability is an question that I cannot fully address here, this should not undermine the general motivation for studying MRP (consider the case of Composition as Identity and Unrestricted Fusion).

Now let us consider whether MRP can lead to a mereology with interesting fusion principles. To start, MRP immediately implies that the universe is "junky".

Proposition 3.1. MRP $\vdash \forall x\exists y(x < y)$

[9]This definition also works for any language that extends $\mathscr{L}^{\propto}_{<}$ with some singular predicates (e.g., $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$). That is, if ψ is an atomic formula with a new singular predicate, $\psi^{<t}$ is just ψ; and the rest is the same. This extended definition will be used in Section 4 and 5.

[10]As before, MRP is really the following axiom scheme.

$$\forall x_0,...,x_n\forall xx_0,...,xx_m(\psi(x_0,...,x_n,xx_0,...,xx_m) \to \exists t\psi^{<t}(x_0,...,x_n,xx_0,...,xx_m)),$$

where ψ is a formula in $\mathscr{L}^{\propto}_{<}$ whose free variables are among $x_0,...,x_n,xx_0,...,xx_m$.

Proof. Fix an a. Applying MRP to $\exists y(y = a)$, we have $\exists t \exists y < t(y = a)$, which is just $\exists t a < t$. Hence, $\forall x \exists y x < y$. □

This then implies that nothing can contain everything as a part given Asymmetry and Transitivity.

Theorem 3.2. $\text{MRP} + \text{Asymmetry} + \text{Transitivity} \vdash \neg\exists x \forall y(y \leqslant x)$

Proof. Suppose there is a u such that $\forall y(y \leqslant u)$. By Proposition 3.1, it follows that there is a v such that $u < v$. Then $v \leqslant u$. By Asymmetry, $v \neq u$ so $v < u$. By Transitivity, $u < u$, which contradicts Asymmetry. □

Unrestricted Fusion implies that there is a fusion of everything, so it is refuted by MRP. A new fusion principle is thus needed. The following principle of *Mereological Separation* serves as a natural replacement of Unrestricted Fusion.

(M-Separation) $xx \neq \emptyset \wedge \exists y(xx \ll y) \rightarrow xx$ fuse.[11]

In words: if xx are a non-empty plurality of proper parts of some y, then xx fuse. M-Separation is simply a localized version of Unrestricted Fusion: it asserts that among things that have bounded by some object in terms of parthood, fusion happens as much as possible. Since Transitivity and Asymmetry are taken for granted, we have arrived at the following new theory of mereology.

Definition 3.5. *Reflective Mereology* (**RM**) the theory that consists of Transitivity, Asymmetry, M-Separation, and all instances of MRP.

3.2.1 Fusion principles in RM

What pluralities fuse in **RM**? Let *Finitary Fusion* be the principle (as a scheme) that for every x_1, ..., x_n, where n is a natural number, there is a fusion of them.

Lemma 3.3. **RM** $\vdash$ Finitary Fusion

[11] Versions of this principle are also discussed in Cotnoir [5] and Bostock [27].

Proof. Fix a_1, ..., a_n. We then have $\exists y(y = a_1) \wedge ... \wedge \exists y(y = a_n)$. By applying MRP to this conjunction, we have that $\exists t(\exists y < t(y = a_1) \wedge ... \wedge \exists y < t(y = a_n))$. This means that $a_1 < t \wedge ... \wedge a_n < t$. By M-Separation, a_1, ..., a_n have a fusion. □

To obtain more interesting fusion principles in **RM**, it is useful to introduce the notion of $<$-absoluteness.

Definition 3.6. A formula ψ is *$<$-absolute* if for any t, $x_1, ..., x_n$ such that $x_1, ..., x_n < t$ and any $yy_1, ..., yy_m$,

$$\psi(x_1, ..., x_n, yy_1, ..., yy_m)^{<t} \leftrightarrow \psi(x_1, ..., x_n, yy_1, ..., yy_m).$$

ψ is *upward $<$-absolute* (*downward $<$-absolute*) if the left-to-right (right-to-left) implication holds.

Intuitively, the truth of an $<$-absolute statement does not depend on how its quantifiers are interpreted. For example, proper parthood, parthood and overlap are all $<$-absolute.

Proposition 3.4. Assume Transitivity. The following formulas are $<$-absolute.

(i) $x \propto yy$, $x < y$, and $x \leqslant y$;

(ii) $x \circ y$.

Proof. Fix t, a, b, cc such that $a, b < t$ and some arbitrary cc.
For (i), $a \propto cc$, $a < b$, and $a \leqslant b$ are $<$-absolute by Definition 3.3.
For (ii), We see that

$$\begin{aligned} a \circ b &\Leftrightarrow \exists z(z \leqslant a \wedge \leqslant b) \\ &\Leftrightarrow \exists z(z < t \wedge z \leqslant a \wedge z \leqslant b) && (\Rightarrow \text{ by Transitivity}) \\ &\Leftrightarrow [\exists z(z \leqslant a \wedge z \leqslant b)]^{<t} \\ &\Leftrightarrow (a \circ b)^{<t} \end{aligned}$$

□

The next lemma is analogous to a weaker version of the Collection Principle in set theory.

Lemma 3.5. Let ψ be an upward $<$-absolute formula with at most two free variables. $\mathbf{RM} \vdash \forall xx[xx \text{ fuse} \wedge \forall x \propto xx \exists z \psi(x,z) \rightarrow \exists t \forall x \propto xx \exists z < t\, \psi(x,z)]$

Proof. Let aa and b be such that $Fu(b,aa)$ and $\forall x \propto aa \exists z \psi(x,z)$. We then have

$$\forall x \propto aa \exists z \psi(x,z) \wedge \exists y\; y = b \tag{1}$$

Applying MRP to (1), it follows that there is some t such that

$$\forall x < t(x \propto aa \rightarrow \exists z < t\, \psi(x,z)^{<t}) \wedge b < t \tag{2}$$

By *Transitivity*, every x among aa is a proper part of t since their fusion b is a proper part of t. So consider any $a \propto aa$. By (2), there is some z such that $z < t$ and $\psi(a,z)^{<t}$; ψ is upward $<$-absolute, so $\psi(a,z)$ and hence t is as desired. □

Theorem 3.6. (M-Replacement^{-})
Let ψ be an upward $<$-absolute formula with at most two free variables.
$\mathbf{RM} \vdash \forall xx \forall yy[(xx \text{ fuse} \wedge \forall x \propto xx \exists! z \psi(x,z) \wedge \forall y \propto yy \exists x \propto xx \psi(x,y)) \rightarrow yy \text{ fuse}]$

Proof. Let aa, bb and c be such that $Fu(c,aa)$ and $\forall x \propto aa \exists! z \psi(x,z) \wedge \forall y \propto bb \exists x \propto aa \psi(x,y)$. We need to show that bb have a fusion. Since ψ is upward $<$-absolute we can apply Lemma 3.5 to get a t such that $\forall x \propto aa \exists z < t\, \psi(x,z)$. If $bb \ll t$, then by M-Separation bb will have a fusion. So it remains to show that $bb \ll t$. Let b be among bb. $\psi(a,b)$ for some a among aa. Then there is some d such that $d < t$ and $\psi(a,d)$. It follows that $b = d$ and hence $b < t$, which completes the proof. □

M-Replacement^{-} can be taken as saying that if xx fuse and there are no more yy than xx in *a certain way*, yy also fuse. A full mereological replacement principle will be proved later when set-theoretic machinery becomes available.

3.2.2 Weak Supplementation

I now turn to the other non-classical facet of **RM**: **RM** turns out to refute Weak Supplementation and Fusion Uniqueness. This will be proved by exploring the consequences of the theory **RM** + Weak Supplementation, which eventually leads to a contradiction. First, I show that under **RM** + Weak Supplementation, some object x will have a mereological complement. i.e., the fusion of all the things that do not overlap x. Second, I observe that the fusion of x and its complement, which exists by Finitary Fusion, contains everything as a part given Weak Supplementation, which contradicts Theorem 3.2.

To begin with, over **RM**, Weak Supplementation and Fusion Uniqueness are equivalent.

Definition 3.7. $Mub(y,xx)$(y is a *minimum upper bound* of xx) $=_{df} xx \leqslant\leqslant y \wedge \forall w(xx \leqslant\leqslant w) \rightarrow y \leqslant w)$

The next lemma says that every fusion is a minimum upper bound under **RM** + Weak Supplementation. We simply adopt the argument in Hovda [13, p. 66] in which only Finitary Fusion is needed.

Lemma 3.7. **RM** $+$ Weak Supplementation $\vdash \forall xx \forall z(Fu(z,xx) \rightarrow Mub(z,xx))$

Proof. Let $Fu(b,aa)$ and suppose that every a in aa is such that $a \leqslant c$ for some c. We need to show that $b \leqslant c$. By Finitary Fusion, some d fuses b and c. Suppose for *reductio* that $d \neq c$. Since $c \leqslant d$, $c < d$. By Weak Supplementation, there is some s such that $s \leqslant d$ and s does not overlap c. Then s must overlap b, so there is some w such that $w \leqslant s$ and $w \leqslant b$. As b fuses aa, w then overlaps some a among aa. $a \leqslant c$ by assumption so by Transitivity w overlaps c. But $w \leqslant s$ so by Transitivity again s overlaps c, which is a contradiction. Therefore, $d = c$ and so $b \leqslant c$. □

Corollary 3.7.1. **RM** $+$ Weak Supplementation $\vdash \forall xx \forall x \forall y(Fu(x,xx) \wedge xx \ll y \rightarrow x \leqslant y)$

Proof. Suppose $Fu(a,aa)$ and $aa \ll b$. By Lemma 3.7, $Mub(a,aa)$ and hence $a \leqslant b$. □

Lemma 3.8. **RM** $\vdash$ Weak Supplementation $\leftrightarrow$ Fusion Uniqueness

Proof. For the left-to-right direction, suppose Weak Supplementation holds and let b and c both fuse some aa. By Lemma 3.7, $Mub(b,aa)$ and $Mub(c,aa)$, so $b \leqslant c \wedge c \leqslant b$, which implies $b = c$ by Asymmetry. For the other direction, suppose Fusion Uniqueness holds and let a and b be such that $a < b$. Suppose for *reductio* that $\forall y \leqslant b(y \circ a)$. Then b fuses a. It follows that $a = b$ since a fuses a, which contradicts Asymmetry. Thus, $\exists y \leqslant b \neg(y \circ a)$. □

The following decomposition principle will be useful later.

(Strong Supplementation) $\forall w(w \leqslant x \rightarrow w \circ y) \rightarrow x \leqslant y$

Strong Supplementation implies Weak Supplementation by Asymmetry. We show that the converse also holds under **RM** (the argument is the same as the one in Hovda [13, pp. 68-69]).

Lemma 3.9. **RM** + Weak Supplementation ⊢ Strong Supplementation

Proof. Let a and b be such that $\forall x(x \leqslant a \rightarrow x \circ b)$. We wish to show that $a \leqslant b$. By Finitary Fusion, some z fuses a and b. Since z fuses z, by Lemma 3.8 it suffices to show that z also fuses b, which will then imply $z = b$ and hence $a \leqslant b$. So it remains to show z fuses b. Clearly, $b \leqslant z$. Consider any w such that $w \leqslant z$. w overlaps a or b. If it overlaps a, then there is some v such $v \leqslant w$ and $v \leqslant a$; v then overlaps b by our assumption, and since $v \leqslant w$ by Transitivity w also overlaps b. Therefore, every part of z overlaps b and so z fuses b, which completes the proof. □

So far only Finitary Fusion, Transitivity and Asymmetry have been used. I now aim to utilize reflection to prove that under **RM** + Weak Supplementation, there is some x such that all the things disjoint from x have a fusion.

Definition 3.8. For any x, dd_x denotes the plurality such that $y \propto dd_x$ iff $\neg(y \circ x)$.

Lemma 3.10. **RM** + Weak Supplementation ⊢ $\exists x\, dd_x \neq \emptyset$

Proof. Proposition 3.1 implies that $x < y$ for some x and y. By Weak Supplementation, there is some $z \leqslant y$ such that $\neg(z \circ x)$, so dd_x are non-empty. □

The next lemma concerns absoluteness, which only uses Transitivity. It says that if x fuses yy^t for some $x < t$, then x fuses yy in t.

Lemma 3.11. Transitivity $\vdash \forall t \forall x < t \forall yy(Fu(x, yy^t) \rightarrow Fu(x, yy)^{<t})$.

Proof. Fix t, $a < t$ and some arbitrary cc. We see that

$$\begin{aligned} Fu(a, cc^t) &\Leftrightarrow \forall x \propto cc^t (x \leqslant a) \wedge \forall y \leqslant a \exists z \propto cc^t (z \circ y) \\ &\Rightarrow \forall x < t(x \propto cc^t \rightarrow x \leqslant a) \wedge \forall y < t(y \leqslant a \rightarrow (\exists z < t(z \propto cc^t \wedge (z \circ y)^{<t})) \\ &\Leftrightarrow (Fu(a, cc))^{<t}. \end{aligned}$$

The second implication holds because "$z \circ y$" is $<$-absolute. □

Lemma 3.12. **RM** + Weak Supplementation $\vdash \forall x(dd_x \neq \emptyset \rightarrow dd_x$ fuse)

Proof. Suppose dd_a are non-empty for some a, i.e., there is some u such that $\neg(u \circ a)$. And suppose for *reductio* that $\neg \exists z Fu(z, dd_a)$. So we have the following

$$\exists y(y = u) \wedge \exists y(y = a) \wedge \neg \exists z Fu(z, dd_a) \tag{3}$$

By MRP there is some t such that

$$u < t \wedge a < t \wedge \neg \exists z < t(Fu(z, dd_a)^{<t}). \tag{4}$$

We claim that there is some z such that $z < t$ and $Fu(z, dd_a^t)$. First note that dd_a^t are non-empty as $u \propto dd_a^t$. Since $dd_a^t \ll t$, by M-Separation, there is some v such that $Fu(v, dd_a^t)$. By Corollary 3.7.1, $v \leqslant t$. But v cannot be identical to t. Otherwise, $a < v$ and hence a overlaps something in dd_a^t, which is impossible. Thus, $v < t$ and $Fu(v, dd_a^t)$. $Fu(v, dd_a^t)$ implies $Fu(v, dd_a)^{<t}$ by Lemma 3.11, so

$$\exists z < t(Fu(z, dd_a)^{<t}). \tag{5}$$

This contradicts (4). Therefore, $\exists z Fu(z, dd_a)$. □

Lemma 3.13. **RM** + Weak Supplementation $\vdash \forall x, y(x \leqslant y \wedge dd_x \leqslant\leqslant y \rightarrow \forall z(z \leqslant y))$

Proof. Let a, b be such that $a \leqslant b$ and $dd_a \leqslant\leqslant b$. Consider any z and any part of z, v. We claim that v must overlap b. Either v overlaps a or not. If it does, since $a \leqslant b$, v overlaps b; if not, then $v \propto dd_a$ so $v \leqslant b$ and hence $v \circ b$. Thus, every part of z overlaps b. By Lemma 3.9, we can then apply Strong Supplementation to get that $z \leqslant b$ for every z. □

Theorem 3.14. **RM** $\vdash \neg$Weak Supplementation $\wedge\, \neg$Fusion Uniqueness

Proof. We show that the theory **RM** + Weak Supplementation is inconsistent. By Lemma 3.10 and Lemma 3.12, there is some a such that dd_a have a fusion b. By *Finitary Fusion*, there is some u that fuses a and b; so $a \leqslant u$ and $dd_a \leqslant u$. By Lemma 3.13, this implies that $x \leqslant u$ for every x, which contradicts Theorem 3.2. This shows that the negation of Weak Supplementation holds in **RM**. By Lemma 3.8, the negation of Fusion Uniqueness also holds in **RM**. □

However, Weak Supplementation does not have to fail *everywhere* in **RM**. In Section 5, I show that it is consistent with **RM** and set theory that classical mereology holds when the quantifiers are restricted to the urelements.

4 Reflective Mereology and Set Theory

In this section, I investigate the interaction between reflective mereology and set theory. We show that a natural strengthening of **RM**, $\mathbf{RM}^+$, implies a full mereological replacement principle in $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$. As a result, $\mathbf{RM}^+$ + Limitation of Size implies that things fuse just in case they form a set. Furthermore, most of the axioms of $\mathbf{ZFCU_2}$ and the existence of various large cardinals can also be derived from $\mathbf{RM}^+$ + Limitation of Size.

We now start to work in the hybrid plural language $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$ (see Section 2.1). **RM**, formulated in $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$, can also talk about sets (e.g., by Proposition 3.1, **RM** implies that every set is a proper part of something). However, a key gradient of reflection principles seems to be missing in MRP in this context. Reflection principles normally assert that every truth is reflected by an *initial segment* of the universe, where an initial segment should contain all the things from which it is built up; but without any further assumptions, it is possible for a set x to be a proper part of y while some member of x is not. This consideration motivates the following definition of initial segment.

Definition 4.1. $IS(t) =_{df} \forall x, y(x < t \wedge y \in x \rightarrow y < t)$

That is, t is an initial segment just in case every member of its proper parts is also a proper part of it. Note that I do not assume that initial segments are transitive sets (nor the other way around). Accordingly, a natural strengthening of MRP is the following.

(MRP$^+$) $\psi \rightarrow \exists t(IS(t) \wedge \psi^{<t})$,

where ψ is in $\mathscr{L}^{\infty}_{<,\in,\mathrm{Ur}}$ (see footnote 9 for the definition of $\psi^{<t}$.)

Definition 4.2. RM$^+$ $=$ Asymmetry $+$ Transitivity $+$ M-Separation $+$ MRP$^+$

Many set-theoretic statements are $<$-absolute for initial segments with respect to mereological relativization.

Proposition 4.1. The following formulas are $<$-absolute (Definition 3.6) for initial segments.

(i) $x = w \cup \{w\}$

(ii) $x = \{w, v\}$

(iii) $x = \langle w, v \rangle$

Proof. Let t be an initial segment and a, b, c be proper parts of t. For (i):

$$\begin{aligned}(a = b \cup \{b\})^{<t} &\Leftrightarrow [b \in a \wedge \forall v \in b(v \in a) \wedge \forall v \in a(v = b \vee v \in b)]^{<t} \\ &\Leftrightarrow b \in a \wedge \forall v < t(v \in b \rightarrow v \in a) \wedge \forall v < t(v \in a \rightarrow v = b \vee v \in b) \\ &\Leftrightarrow b \in a \wedge \forall v \in b(v \in a) \wedge \forall v \in a(v = b \vee v \in b) \\ &\Leftrightarrow a = b \cup \{b\}\end{aligned}$$

The second line implies the third line because t is an initial segment.

For (ii):

$$\begin{aligned}(a = \{b, c\})^{<t} &\Leftrightarrow [b \in a \wedge c \in a \wedge \forall x \in a(x = b \vee w = c)]^{<t} \\ &\Leftrightarrow b \in a \wedge c \in a \wedge \forall x < t(x \in a \rightarrow x = b \vee w = c)\end{aligned}$$

$$\Leftrightarrow b \in a \wedge c \in a \wedge \forall x(x \in a \rightarrow x = b \vee w = c)$$
$$\Leftrightarrow a = \{b, c\}$$

For (iii),

$$\begin{aligned}
(a = \langle b,c \rangle)^{<t} &\Leftrightarrow [\exists w \in a(w = \{b\}) \wedge \exists v \in a(v = \{b,c\}) \wedge \forall v \in a(v = \{b\} \vee v = \{b,c\})]^{<t} \\
&\Leftrightarrow (\exists w \in a(w = \{b\}))^{<t} \wedge (\exists v \in a(v = \{b,c\}))^{<t} \wedge (\forall v \in a(v = \{b\} \vee v = \{b,c\})^{<t} \\
&\Leftrightarrow \exists w \in a(w = \{b\}) \wedge \exists v \in a(v = \{b,c\}) \wedge \forall v \in a(v = \{b\} \vee v = \{b,c\}) \\
&\Leftrightarrow a = \langle b,c \rangle
\end{aligned}$$

The third equivalence holds because t is initial and "$x = \{w, v\}$" is $<$-absolute. □

Lemma 4.2. Let ff be a plurality. For any initial segment t and any $x, y < t$, if $(\langle x,y \rangle \propto ff)^{<t}$, then $\langle x,y \rangle \propto ff$.

Proof. Let t be initial and a, b be proper parts of t.

$$\begin{aligned}
(\langle b,c \rangle \propto ff)^{<t} &\Leftrightarrow [\exists w \propto ff(w = \langle b,c \rangle)]^{<t} \\
&\Leftrightarrow \exists w < t(w \propto ff^t \wedge (w = \langle b,c \rangle)^{<t}) \\
&\Rightarrow \exists w(w \propto ff \wedge (w = \langle b,c \rangle) \\
&\Leftrightarrow \langle b,c \rangle \propto ff
\end{aligned}$$

The third implication holds because $(w = \langle b,c \rangle)^{<t}$ implies $w = \langle b,c \rangle$ for $w < t$, as shown in the previous proposition. □

These absoluteness results enable us to prove the full mereological replacement: if *xx* fuse and there are no more *yy* than *xx*, then *yy* also fuse.

Theorem 4.3. $\mathbf{RM}^+ \vdash \forall xx, yy(xx \text{ fuse} \wedge yy \preceq xx) \rightarrow yy \text{ fuse}$

Proof. The idea is exactly the same as in the proof of Theorem 3.6. Let *aa* and *bb* be such that *aa* is fused by c and $bb \preceq aa$. Then there is some ff such that $ff : aa \xrightarrow{\text{onto}} bb$. So we have

$$\forall x \propto aa \exists z(\langle x,z\rangle \propto ff) \wedge \exists y(y=c) \tag{6}$$

By applying MRP^+ to it, we see that there is some initial segment t such that

$$\forall x < t[x \propto aa \to \exists z < t(\langle x,z\rangle \propto ff)^{<t}] \wedge c < t \tag{7}$$

Since c fuses aa, $aa \ll t$. So for every $x \propto aa$, there is some $z < t$ such that $(\langle x,z\rangle \propto ff)^{<t}$. By Lemma 4.2, this implies that $\langle x,z\rangle \propto ff$. Thus, we have

$$\forall x \propto aa \exists z < t(\langle x,z\rangle \propto ff) \tag{8}$$

As before, to show bb have a fusion it suffices to show that $bb \ll t$ given M-Separation. Consider any b among bb. Since ff is onto, there is some $a \propto aa$ such that $\langle a,b\rangle \propto ff$. But then b must be a proper part of t as ff is a function. □

The situation with Limitation of Size also becomes different. $\mathbf{RM}^+$ and LS jointly imply the following principle, which says that a non-empty plurality of things fuse just in case they form a set.

(Limitation of Fusion) $xx \neq \emptyset \to (xx \text{ fuse} \leftrightarrow xx \text{ form a set})$

Theorem 4.4. $\mathbf{RM}^+ + \text{LS} \vdash \text{Limitation of Fusion}$

Proof. We first show that every non-empty plurality form a set only if they have a fusion. Let s be the set of aa, which are non-empty. By the same proof in Proposition 3.1, we see that $s < t$ for some initial segment t and so $aa \ll t$. By M-Separation, aa have a fusion.

For the other direction, let aa fuse and suppose for *reductio* that $uu \preceq aa$. By Theorem 4.3, uu fuse, which contradicts Theorem 3.2. Hence, $uu \npreceq aa$ so aa form a set by LS. □

Furthermore, the theory $\mathbf{RM}^+ + \text{Limitation of Size} + \text{Extensionality}$ also provides a succinct axiomitization of $\mathbf{ZFCU_2} - \text{Foundation}$.

Theorem 4.5. $\mathbf{RM}^+ + \text{LS} + \text{Extensionality} \vdash \mathbf{ZFCU_2} - \text{Foundation}$

Proof. Empty Set. Let aa be the empty plurality, Clearly, there is no onto function from aa to uu. So aa form a set, which is the empty set.

Pairing. For any a and b, by *Finitary Fusion*, we know that they fuse. By Limitation of Fusion, $\{a,b\}$ exists.

Union. Let s be a set and aa_s be such that $x \propto aa_s$ iff $\exists y(y \in s \wedge x \in y)$. We may assume that aa_s are not empty. By MRP^+, s is a proper part of some initial segment t. But then aa_s are proper parts of t: if $x \in y$ and $y \in s$, then $y < t$ so $x < t$. By M-Separation, aa_s fuse and by Limitation of Fusion, the aa_s form a set which is the union of s.

$\in$-Separation. Let bb form a set and $aa \propto\propto bb$. By Limitation of Size, there is no surjective map from bb onto everything. Then there is no surjective map from aa onto everything and hence aa form a set by Limitation of Size again.

$\in$-Replacement. Let s be the set of bb and suppose that $aa \preceq bb$. bb then have a fusion by Limitation of Fusion; then by 4.3, aa form a fusion so aa form a set by Limitation of Fusion again.

Infinity. By Union and Pairing, it follows that

$$\forall x \exists y(y = x \cup \{x\}) \wedge \exists z(z = 0), \tag{9}$$

where 0 is the empty set. By MRP^+, there is some initial segment t such that

$$\forall x < t \exists y < t(y = x \cup \{x\})^{<t} \wedge 0 < t \tag{10}$$

By Lemma 4.1, it follows that

$$\forall x < t \exists y < t(y = x \cup \{x\}). \tag{11}$$

Now let pp_t be the proper parts of t that are sets. The proper parts of t form a set because they fuse; so by $\in$-Separation, pp_t must also form a set. Let s be the set of pp_t. s is inductive: clearly, $0 \in s$; if $x \in s$, then $x \cup \{x\}$ is in s.

Power Set. Let s be a set and xx_s be its members. By $\in$-Separation we have

$$\forall yy \propto\propto xx_s(yy \text{ form a set}) \wedge \exists y(y = s) \tag{12}$$

Applying MRP^+ to it gives us an initial segment t such that

$$\forall yy \ll t[yy \propto\propto xx_s^t \to (\exists z \forall v(v \in z \leftrightarrow v \propto yy))^{<t}] \wedge s < t \tag{13}$$

Since t is initial, $xx_s \ll t$. So we have

$$\forall yy \propto\propto xx_s \exists z < t \forall v(v \in z \leftrightarrow v \propto yy) \tag{14}$$

Now let ww be the plurality such that $w \propto ww$ iff $w < t$ and w is a *subset* of s. By M-Separation and Limitation of Fusion, ww form a set v. By Extensionality, v is the power set of s.

Choice. By a standard argument, Limitation of Size implies that there exists a bijection between the plurality of the ordinals and the universe. So every set can be well-ordered. This completes the proof. □

$\mathbf{RM}^+ +$ Limitation of Size also implies the existence of large cardinals because it implies the set-theoretic reflection.

Proposition 4.6. $\mathbf{RM}^+ + \mathrm{LS} \vdash \mathrm{RP}_2$

Proof. (Sketch) Let ψ be a formula in $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$. Suppose ψ holds. Then by MRP^+, there is some initial segment t such that $\psi^{<t}$. Let s be the set of the proper parts of t, which exists by Limitation of Fusion. s is transitive: if $x \in y$ and $y \in s$, $y < t$ so $x < t$ and hence $x \in s$. By an easy induction on ψ, it follows that $\psi^{<t}$ is equivalent to $\psi^{\in s}$ (see Definition 3.1). Therefore, there is a transitive set s such that $\psi^{\in s}$. □

It is known that $\mathbf{ZFC_2}$ + RP_2 implies the existence of proper-class many inaccessible cardinals, Mahlo cardinals, and weakly-compact cardinals (see Tait [24]). And it is proved in Yao[32] that $\mathbf{ZFCU_2}$ + RP_2 + LS and $\mathbf{ZFC_2}$ + RP_2 are mutually interpretable and that as a consequence, $\mathbf{ZFCU_2}$ + RP_2 + LS also yields the existence of these large cardinals. This concludes our discussion of how reflective mereology and set theory interact.[12]

[12] A further question to explore is whether **RM** (or some pure mereology extending **RM**) can serve as a foundation of mathematics by itself given that classical mereology, as shown in Hamkins and Kikuchi [11], falls short on this task.

5 A Model of $\mathbf{RM}^+$ + $\mathbf{ZFCU_2}$

In this section, I provide a natural interpretation of the hybrid theory $\mathbf{RM}^+$ + Atomicity + $\mathbf{ZFCU_2}$ + LS in $\mathbf{ZFCU_2}$ + RP_2. Since the latter theory can be shown to be consistent relative to $\mathbf{ZFC_2}$ + RP_2[13], this shows that reflective mereology is consistent with set theory. Also, our interpretation of proper parthood shows that it is consistent with $\mathbf{RM}^+$ + Atomicity (and set theory) that parthood behaves classically on the urelements.

We start in $\mathbf{ZFCU_2} + \mathrm{RP}_2$ + LS with an assumption that the urelements form a set of size 2^κ for some infinite cardinal κ. We then extend the language $\mathscr{L}^{\infty}_{\in,\mathrm{Ur}}$ by adding a constant symbol $\mathbf{f}$. Let $A_\mathbf{f}$ be the axiom that $\mathbf{f}$ is a bijection from the set of urelements to $P(\kappa)\setminus\{\emptyset\}$.

Definition 5.1. $\mathbf{ZFCU}_2^+ = \mathbf{ZFCU_2}$ + LS + $A_\mathbf{f}$.

Definition 5.2. $\mathbf{ARM}^+ = \mathbf{RM}^+$ + Atomicity

$\mathbf{ZFCU}_2^+ + \mathrm{RP}_2$ will serve as the base theory within which $\mathbf{ARM}^+$ is interpreted.[14] With $\mathbf{f}$ we can simulate a classical proper parthood relation on the urelements. We say that x is a "classical proper part" of y just in case $\mathbf{f}(x) \subsetneq \mathbf{f}(y)$. For any object x, let $TC(x)$ denote the transitive closure of x, i.e., $TC(x)$ is the smallest transitive set t such that every member of x is in t. $\mathbf{ZFCU}_2^+$ certainly proves that everything has a transitive closure (the transitive closure of any urelement is the empty set). Then x is said to be a "proper part" of y just in case either x is a classical proper part of y, or x is a *classical part* of something in the transitive closure of y. Formally,

Definition 5.3. $x \sqsubset y =_{df} \exists v, w(v = \mathbf{f}(x) \wedge w = \mathbf{f}(y) \wedge v \subsetneq w)$
$x \sqsubseteq y =_{df} x \sqsubset y \vee x = y$
$x \lhd y =_{df} x \sqsubset y \vee \exists z(x \sqsubseteq z \wedge z \in TC(y))$
$x \unlhd y =_{df} x \lhd y \vee x = y$

[13] See Theorem 3.3 in Yao [32]

[14] Another way to think about this is to start with a model $\mathscr{M} = \langle M, P(M), \in^{\mathscr{M}}, \mathrm{Ur}^{\mathscr{M}}\rangle$ of $\mathbf{ZFCU_2} + \mathrm{RP}_2$ with 2^κ many urelements and then in M, we fix a bijection f from the set of urelements to $P(\kappa)\setminus\{\emptyset\}$ as a parameter. Now I proceed to produce a model $\mathscr{N} = \langle M, P(M), \in^{\mathscr{M}}, \mathrm{Ur}^{\mathscr{M}}, <^{\mathscr{N}}\rangle$ of $\mathbf{RM}^+$ + Atomicity + $\mathbf{ZFCU_2}$ + LS.

$\lhd$ is a definable relation in $\mathbf{ZFCU}_2^+ + \mathrm{RP}_2$ and will be the intended interpretation of proper parthood. We then define an interpretation σ from $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$ to $\mathscr{L}^{\propto}_{\in,\mathrm{Ur},\mathbf{f}}$

Definition 5.4. Let ψ be in $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$. We define ψ^σ inductively as follows.

If ψ is atomic but not some $x < y$, ψ^σ is ψ;

if ψ is $x < y$, ψ^σ is $x \lhd y$;

if ψ is $\chi \vee \varphi$, ψ^σ is $\chi^\sigma \vee \varphi^\sigma$;

if ψ is $\neg\varphi$, ψ^σ is $\neg\varphi^\sigma$;

if ψ is $\exists x\varphi$, ψ^σ is $\exists x\varphi^\sigma$;

if ψ is $\exists xx\varphi$, ψ^σ is $\exists xx\varphi^\sigma$.

σ clearly preserves logical consequence. And trivially, if $\psi \in \mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$ is an axiom of $\mathbf{ZFCU}_2$ + LS, then ψ^σ is provable in $\mathbf{ZFCU}_2^+ + \mathrm{RP}_2$. So to show that $\mathbf{ZFCU}_2$ + LS + $\mathbf{ARM}^+$ is consistent realtive to $\mathbf{ZFCU}_2^+ + \mathrm{RP}_2$, it suffices to show that if ψ is an axiom of $\mathbf{ARM}^+$, $\mathbf{ZFCU}_2^+ + \mathrm{RP}_2$ will prove ψ^σ, i.e., all the axioms of $\mathbf{ARM}^+$ hold in $\mathbf{ZFCU}_2^+ + \mathrm{RP}_2$ when proper parthood is interpreted as $\lhd$.

We unpack some interpreted formulas and axioms for reference.

$(x \circ y)^\sigma =_{df} \exists z(z \unlhd x \wedge z \unlhd y)$

$Atom(x)^\sigma =_{df} \forall y(y \not\lhd x)$

$Fu(x,yy)^\sigma =_{df} \forall y \propto yy(y \unlhd x) \wedge \forall z \unlhd x \exists y \propto yy(y \circ z)^\sigma$

Transitivity$^\sigma =_{df} \forall x,y,z(x \lhd y \wedge y \lhd z \to x \lhd z)$

Asymmetry$^\sigma =_{df} \forall x,y(x \lhd y \to y \not\lhd x)$

Atomicity$^\sigma =_{df} \forall x \exists y(y \unlhd x \wedge Atom(y)^\sigma)$

M-Separation$^\sigma =_{df} \forall xx(xx \neq \emptyset \wedge \exists y(\forall x \propto xx(x \lhd y)) \to \exists z Fu(z,xx)^\sigma)$

.

Proposition 5.1. ($\mathbf{ZFCU}_2^+$)

For any x and y,

(i) if x, y are urelements, $a \lhd b$ iff $a \sqsubset b$;

(ii) if x, y are sets, $x \lhd y$ iff $x \in TC(y)$;

(iii) if x is a set and y is a urelement, $x \ntriangleleft y$;

(iv) if $x \in y$, then $x \lhd y$;

(v) if $x \in TC(\{y\})$, then $(x \circ y)^\sigma$.

(vi) if x is a urelement, then there is some urelement z such that $z \trianglelefteq x \wedge Atom(z)^\sigma$.

Proof. (i) holds because because the transitive closure of any urelement is empty. The left-to-right direction of (ii) holds because $x \sqsubset y$ only if both x and y are urelements. (iii) holds for the same reason. (iv) holds since $x \in y$ implies $x \in TC(y)$. For (v), suppose that $x \in TC(\{y\})$. Then either $x = y$ or $x \in TC(y)$ so we may assume $x \in TC(y)$; so $x \lhd y$, which implies $(x \circ y)^\sigma$. For (vi), if x is a urelement then $\mathbf{f}(x)$ is a non-empty subset of κ. Fix $\alpha \in \mathbf{f}(x)$. There is some urelement z such that $\mathbf{f}(z) = \{\alpha\}$ and so $z \sqsubseteq x$; but then $z \trianglelefteq x$ and clearly, $Atom(z)^\sigma$. □

From 5.1 (i), it follows that $\lhd$ behaves classically on the urelements as $\subsetneq$ behaves the same as classical proper parthood on $P(\kappa) \setminus \{\emptyset\}$. More precisely, for any non-empty plurality of urelements xx, there is a unique urelement y such that $Fu(y, xx)^\sigma$.

Proposition 5.2. $\mathbf{ZFCU}_2^+ \vdash \forall xx[(xx \neq \emptyset \wedge \forall x \propto xx\, \mathrm{Ur}(x)) \rightarrow \exists! y(\mathrm{Ur}(y) \wedge Fu(y, xx)^\sigma)]$

Proof. Let aa be a non-empty plurality of urelements. Since in $\mathbf{ZFCU}_2^+$ the urelements form a set, aa form a set A. Let $z =: \bigcup_{a \in A} \mathbf{f}(a)$, which is a non-empty subset of κ. Then there is some urelement y such that $\mathbf{f}(y) = z$. It is then easy to check that $Fu(y, aa)^\sigma$ and the existence such urelement y is unique. □

Similarly, it can be easily seen that Weak Supplementation$^\sigma$ also holds when the quantifiers are restricted to the urelements. That is, if x and y are urelements and $x \lhd y$, then there is some urelement z such that $z \trianglelefteq y$ and $\neg(z \circ x)^\sigma$.

Lemma 5.3. $\mathbf{ZFCU}_2^+ \vdash \text{Asymmetry}^\sigma \wedge \text{Transitivity}^\sigma \wedge \text{Atomicity}^\sigma$

Proof. For Asymmetry$^\sigma$, suppose that $x \lhd y$. By 5.1 (iii), we may assume that either x, y are both sets, or x, y are both urelements. In the former case, $y \not\lhd x$ by Foundation; in the latter, $y \not\lhd x$ by the definition of $\sqsubset$.

For Transitivity$^\sigma$, suppose that $x \lhd y$ and $y \lhd z$. If either they are all sets or they are all urelements, we will have $x \lhd z$ by definition of $\lhd$ and 5.1 (ii). If not, only two possibilities remain by 5.1 (iii): either (a) x is a urelement, while y and z are sets, or (b) z is a set, while x and y are urelements. If (a) holds, it follows that there is a k such that $x \sqsubseteq k \wedge k \in TC(y)$ and $y \in TC(z)$; so $k \in TC(z)$ and hence $x \lhd z$. In case (b), $x \sqsubset y$, and there is some k such that $y \sqsubseteq k \wedge k \in TC(z)$; then $x \sqsubseteq k$ and so $x \lhd z$.

For Atomicity$^\sigma$, fix an x. By 5.1 (vi), we may assume x is a set. If $TC(x)$ contains a urelement a, then by 5.1 (vi) again there is some urelement b such that $Atom(b)^\sigma$ and $b \unlhd a$, which implies $b \unlhd x$. If x is a pure set, then $\emptyset \unlhd x$ and $Atom(\emptyset)^\sigma$. □

We next verify that $\mathbf{ZFCU}_2^+ + \text{RP}_2$ proves M-Separation$^\sigma$. Facts in Proposition 5.1 will be used without further mention.

Lemma 5.4. $\mathbf{ZFCU}_2^+ \vdash \forall y \forall xx(xx \neq \emptyset \wedge y \text{ is a set of } xx \rightarrow Fu(y, xx)^\sigma)$

Proof. Let y be the set of xx. Then for every $x \propto xx$, $x \in y$ and hence $x \unlhd y$. Consider any z such that $z \lhd y$. Then there is some k such that $z \sqsubseteq k \wedge k \in TC(y)$. We wish to show that z "overlaps" some x among xx. Suppose that z is a urelement. Since $k \in TC(y)$, $k \in TC(\{x\})$ for some $x \propto xx$ and so $(k \circ x)^\sigma$; clearly, $z \unlhd k$ and by Transitivity$^\sigma$, it follows that $(z \circ x)^\sigma$. If z is a set, then we have $z \in TC(y)$ and so $z \in TC(\{x\})$ for some $x \propto xx$ and hence $(z \circ x)^\sigma$. Therefore, $Fu(y, xx)^\sigma$. □

Lemma 5.5. $\mathbf{ZFCU}_2^+ \vdash \text{M-Separation}^\sigma$

Proof. Let y and xx be such that $\forall x \propto xx(x \lhd y)$, where xx are non-empty. We first observe that xx form a set. Define s to be

$$s = TC(\{y\}) \cup \{v : \text{Ur}(v) \wedge \exists k(v \sqsubseteq k \wedge k \in TC(\{y\}))\}$$

s is a set by $\mathbf{ZFCU}_2^+$ and the fact that the urelements form a set, and it is clear that xx are members of s by checking the definition of $\lhd$. So by $\in$-Separation it follows that xx form a set z and hence $Fu(z,xx)^\sigma$ by the previous lemma, □

Finally, let us turn to MRP$^\sigma$.

Definition 5.5. x is $\sqsubset$-closed $=_{df} \forall y,z(y \in x \wedge z \sqsubset y \rightarrow z \in x)$

Proposition 5.6. If t is a transitive set that is $\sqsubset$-closed, then for every x, $x \lhd t$ if and only if $x \in t$

Proof. Suppose $x \lhd t$. If x is a set, $x \in TC(t)$ so $x \in t$ as t is transitive. If x is a urelement, then for some k, $x \sqsubseteq k \wedge k \in TC(t)$; so $k \in t$ and hence $x \in t$ because t is $\sqsubset$-closed. The other direction is by 5.1 (iv). □

Transitive and $\sqsubset$-closed sets that are sufficiently tall will have the following nice property.

Lemma 5.7. ($\mathbf{ZFCU}_2^+$) Let t be such (i) t is transitive and $\sqsubset$-closed; (ii) $\mathbf{f} \in t$; (iii) t is sufficiently tall.[15] For any $\psi \in \mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$, any $x_0,...,x_i$ and any $xx_0,...,xx_j$,

$$[\psi(x_0,...,x_i,xx_0,...,xx_j)^\sigma]^{\in t} \leftrightarrow [\psi(x_0,...,x_i,xx_0,...,xx_j)^{<t}]^\sigma$$

Proof. The proof is by induction on the complexity of ψ.
Atomic cases. Suppose ψ is $x \propto xx$. Then $(\psi^\sigma)^{\in t} = (x \propto xx)^{\in t} = x \propto xx \cap t$. But,

$$\begin{aligned} x \in xx \cap t &\Leftrightarrow x \propto xx \wedge x \in t \\ &\Leftrightarrow x \propto xx \wedge x \lhd t \\ &= (x \propto xx \wedge x < t)^\sigma \\ &= ((x \propto xx)^{<t})^\sigma \end{aligned}$$

[15] The height of any transitive set t is measured by $Ord \cap t$. Here, "t is sufficiently tall" can be understood as "the full second-order model $\langle t, P(t), \in \rangle$ satisfies $\mathbf{ZFCU}_2$", in which case the height of t must be an inaccessible cardinal.

The second equivalence holds because t is transitive and $\sqsubset$-closed. Hence, $(\psi^{\sigma})^{\in t} \leftrightarrow (\psi^{<t})^{\sigma}$[16].

Suppose ψ is $x < y$. Then ψ^{σ} is $x \lhd y$. Since t is transitive, sufficiently tall and contains **f**, "$x \lhd y$" is absolute for t, i.e., $(x \lhd y)^{\in t}$ iff $x \lhd y$. But $x \lhd y$ is $[(x < y)^{<t}]^{\sigma}$. Therefore, $(\psi^{\sigma})^{\in t} \leftrightarrow (\psi^{<t})^{\sigma}$.

Inductive steps. Suppose ψ is $\chi \vee \varphi$. Then

$$\begin{aligned}(\psi^{\sigma})^{\in t} &= (\chi^{\sigma})^{\in t} \vee (\varphi^{\sigma})^{\in t} \\ &\Leftrightarrow (\chi^{<t})^{\sigma} \vee (\varphi^{<t})^{\sigma} && \text{(induction hypothesis)} \\ &= [(\chi \vee \varphi)^{<t}]^{\sigma} \\ &= (\psi^{<t})^{\sigma}.\end{aligned}$$

Suppose ψ is $\neg\chi$. Then

$$\begin{aligned}(\psi^{\sigma})^{\in t} &= \neg[(\chi^{\sigma})^{\in t}] \\ &\Leftrightarrow \neg[(\chi^{<t})^{\sigma}] && \text{(induction hypothesis)} \\ &= [(\neg\chi)^{<t}]^{\sigma} \\ &= (\psi^{<t})^{\sigma}.\end{aligned}$$

Suppose ψ is $\exists x\varphi$.

$$\begin{aligned}(\psi^{\sigma})^{\in t} &= (\exists x\varphi^{\sigma})^{\in t} \\ &= \exists x(x \in t \wedge (\varphi^{\sigma})^{\in t}) \\ &\Leftrightarrow \exists x(x \lhd t \wedge (\varphi^{<t})^{\sigma}) \\ &= \exists x[(x < t)^{\sigma} \wedge (\varphi^{<t})^{\sigma}] \\ &= ((\exists x\varphi)^{<t})^{\sigma} \\ &= (\psi^{<t})^{\sigma}.\end{aligned}$$

[16] For the last identity, it is a trivial observation that nothing will be affected if we let $(x \propto xx)^{<t}$ just be $x \propto xx \wedge x < t$. So I shall adopt this way of relativization here.

The second equivalence holds by induction hypothesis and the fact that t is transitive and $\sqsubset$-closed. Finally, suppose ψ is $\exists xx\varphi$. By the same reasoning, we will have

$$\begin{aligned}(\psi^{\sigma})^{\in t} &= \exists xx(xx \in\in t \wedge (\varphi^{\sigma})^{\in t}) \\ &\Leftrightarrow \exists xx(\forall y(y \propto xx \rightarrow y \lhd t) \wedge (\varphi^{<t})^{\sigma}) \\ &= \exists xx[(xx \ll t)^{\sigma} \wedge (\varphi^{<t})^{\sigma}] \\ &= ((\exists xx\varphi)^{<t})^{\sigma} \\ &= (\psi^{<t})^{\sigma}.\end{aligned}$$

□

Theorem 5.8. For every axiom φ of $\mathbf{ARM}^{+}$, $\mathbf{ZFCU}_2^{+} + \mathrm{RP}_2 \vdash \varphi^{\sigma}$.

Proof. In view of Lemma 5.3 and 5.5, it remains to verify that if φ is an instance of MRP^{+}, φ^{σ} holds. Any such φ will be of the following form.

$$\forall x_0,...,x_i \forall xx_0,...,xx_j[\psi \rightarrow \exists t(IS(t) \wedge \psi^{<t})]. \tag{15}$$

Thus, we need to show that

$$\forall x_0,...,x_i \forall xx_0,...,xx_j[\psi^{\sigma} \rightarrow \exists t(IS(t)^{\sigma} \wedge (\psi^{<t})^{\sigma})]. \tag{16}$$

So fix $x_0,...,x_i,xx_0,...,xx_j$ and suppose that ψ^{σ} holds. Let u be the set of all urelements. Then we have

$$\psi^{\sigma} \wedge \exists x(x = \mathbf{f}) \wedge \exists x(x = u) \wedge \mathrm{ZFCU}_2. \tag{17}$$

where ZFCU_2 is the conjunction of all the axioms of $\mathbf{ZFCU}_2$. By RP_2, it follows that there is a transitive set t such that

$$(\psi^{\sigma})^{\in t} \wedge \mathbf{f} \in t \wedge u \in t \wedge (\mathrm{ZFCU}_2)^{\in t}. \tag{18}$$

t is sufficiently tall since it satisfies all the axioms of $\mathbf{ZFCU}_2$. t is $\sqsubset$-closed for the trivial reason that all the urelements are in t. Lemma 5.7 thus applies, so we have

$$(\psi^{<t})^{\sigma}. \tag{19}$$

It is easy to see that $IS(t)^{\sigma}$ holds (see Definition 4.1). For, if $z \in y$ and $y \lhd t$, then y is a set so $y \in TC(t)$; so $z \in TC(t)$ and hence $z \lhd t$. Therefore,

$$\exists t(IS(t)^{\sigma} \wedge (\psi^{<t})^{\sigma}). \tag{20}$$

This completes the proof. □

6 Some Philosophical Remarks

Let us consider whether **RM** satisfies the two desiderata proposed in Section 2. Namely, a well-motivated mereology should

> (i) have fusion principles that are compatible with the existence of ordinary objects and (ii) be motivated by independent philosophical considerations.

For (ii), the two novel axioms of **RM**, as noted earlier, both have independent philosophical motivations. MRP can be seen a way of articulating the metaphysical thesis that the universe of everything is indescribable, which is a natural generalization of the Cantorian conception of set. M-Separation is guided by one metaphysical principle underlying Unrestricted Fusion—there should be no arbitrary restriction on when things have a mereological fusion. And since Uzquiano's cardinality problem shows that Unrestricted Fusion takes this principle too far in the presence of certain set-theoretic axioms, M-Separation seems to be a natural way of weakening Unresricted Fusion when the domain of quantification includes everything. The situation here is thus analogous to using $\in$-Separation as a replacement of the Naive Comprehension Principle in set theory in response to Russell's Paradox.

For (i), while **RM** refutes Unrestricted Fusion, it implies Finitary Fusion and a weaker version of Mereological Replacement, which allows "small" pluralities to fuse. So **RM** has no difficulty of allowing ordinary objects to exist. Furthermore, $\mathbf{RM}^+$ implies the full Mereological Replacement; and with Limitation of Size, it implies Limitation of Fusion. In fact, Limitation of Fusion has been brought up as an alternative fusion principle in many places (see Rosen [21] and Uzquiano [28]), yet it was often dismissed eventually due to the lack of independent justification. Within the framework of reflective mereology, however, Limitation of Fusion follows as a theorem. Suffice it to say that reflective mereology has intuitive and interesting fusion principles.

The similarity between fusion and set formation under reflective mereology may draw some skepticism, especially in view of the somewhat surprising result that **RM** refutes Weak-Supplementation and Fusion Uniqueness (see Theorem 3.14). One might wonder: why should $<$ in **RM** be interpreted as proper parthood after all?

Firstly, we know that any mereology that avoids Uzquiano's cardinality problem has to differ with **CM** in some respect. And whether Weak Supplementation should be seen as an analytic truth about parthood is, in fact, a controversial issue (for recent discussions on this matter, see Donnelly [7] and Cotnoir [6]). While the *prima facie* plausibility of Weak Supplementation may come from our everyday experiences with ordinary objects, its failure in **RM** suggests that this decomposition principle should not be taken as an *absolutely general* principle. Also, one should note that reflective mereology does not force Weak Supplementation to fail everywhere: as we have seen in the previous section, there is a model of reflective mereology and set theory where $<$ behaves classically on the urelements. However, reflective mereology does not impose any substantial mereological structure on the urelements either,[17] which allows one to adopt their favorite mereological structure on urelements.

Secondly, the fact that $<$ behaves similarly as "in the transitive closure of" (and hence fusion behaves similarly as set formation) is not a good reason to deny that $<$ deserves its mereological reading. On the contrary, there is a natural sense in which things in the transitive closure of a set are parts of the set since a set is "made up" precisely by things in its transitive closure. Indeed, this is part of the intuition behind Definition 5.3, which can be stated as follows.

[17]There can be models of $\mathbf{RM}^+$ and set theory where the urelements do not have any interesting mereological structure.

(Set-Part) $Set(s) \to \forall x(x < s \leftrightarrow \exists z \in TC(s)(x \leqslant z))$

In words: x is a proper part of a set s if and only if, x is part of something in the transitive closure of s. According to Set-Part, Socrates and all the atoms that compose Socrates will all be proper parts of $\{$Socrates$\}$.[18] Set-Part fits nicely with reflective mereology. For instance, given Set-Part everything is an initial segment (Definition 4.1) and hence **RM** becomes equivalent to $\mathbf{RM}^{+}$. Also, by the same argument as in Lemma 5.4, one can show that, under Set-Part, sets are fusions of their members. The model constructed in the last section indeed satisfies Set-Part and serves as a natural model of $\mathbf{RM}^{+}$.[19]

[18]This seems to have an bizarre consequence that $\{$Socrates$\}$, as an abstract object, has concrete objects as its parts. However, the same situation can happen whenever Finitary Fusion holds: the fusion of $\emptyset$ and Socrates is arguably abstract but it has Socrates a part. In any case, Set-Part is not part of reflective mereology but only a complementary principle one might consider to adopt. I thank an anonymous referee for raising this worry.

[19]Notably, Set-Part differs from Lewis' principle in [17] that the parts of a non-empty set are all and only its non-empty subsets. It is not known if Lewis' principle is consistent with **RM** + $\mathbf{ZFCU}_2$ + LS.

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