Reflection with Absolute Generality

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Reflection Principles in Pure Set Theory

The iterative conception of set

$V_0 = \emptyset$;

$V_{\alpha+1} = P(V_\alpha)$;

$V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$, where γ is a limit;

$V = \bigcup_{\alpha<Ord} V_\alpha$.

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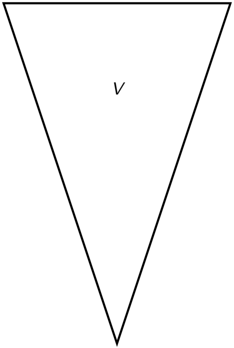

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The Lévy-Montague Reflection Principle holds in ZF.

Remark. For this reason, first-order reflection is seen as a consequence of the iterative conception of set.

Second-order reflection

Bernays' Reflection

(RP_2) $\forall X[\varphi(X) \to \exists t(t \text{ is transitive} \wedge \varphi^t(X \cap t))]$, where φ is any formula in the language of class theory.

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An ω-Erdős cardinal is consistent with V=L, so RP_2 is a weak large cardinal axiom.

Set Theory with Absolute Generality

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ZFC is not an absolutely general theory: it assumes that everything is a set.

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Zermelo (1930) actually considered set theory with a class of urelements.

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Let A be a *set* of urelements.

$V_0(A) = A$;
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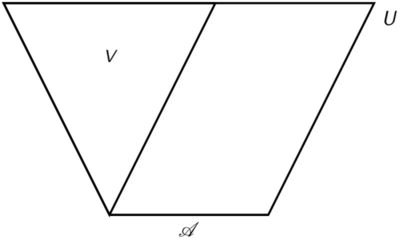

Question

How do reflection principles behave in absolute generality?

First-Order Reflection

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Partial reflection: any true statement is true in some transitive set containing the parameters.

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Are RP and RP^- provable from "urelement set theory"? Are they equivalent?

ZFU_R

Definition

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Definition

$\mathsf{ZFU_R} = \mathsf{ZU}$ + Replacement.
$\mathsf{ZFCU_R} = \mathsf{ZFU_R}$ + AC.
$\mathsf{ZF} = \mathsf{ZFU_R} + \forall x \neg \mathcal{A}(x)$.
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Note. The subscript R indicates that we are only working with Replacement.

Interpreting U in V

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Definition (Barwise?)

Let $\langle V, \in \rangle$ be a model of ZF and X be a class of V. In V, define by recursion

$$V[\![X]\!] = (\{0\} \times X) \cup \{\bar{x} \in V : \exists x(\bar{x} = \langle 1, x \rangle \wedge x \subseteq V[\![X]\!])\}.$$

For every $\bar{x}, \bar{y} \in V[\![X]\!]$,

$$\bar{x} \mathrel{\bar{\in}} \bar{y} \text{ iff } \exists y(\bar{y} = \langle 1, y \rangle \wedge \bar{x} \in y);$$

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$\bar{\mathscr{A}}(\bar{x})$ iff $\bar{x} \in \{0\} \times X$.

Theorem

Let V be a model of ZF and X be a class of V. Then

- $V[\![X]\!] \models \mathrm{ZFU}_R$;
- $V[\![X]\!] \models AC$ *iff* $V \models AC$;
- $\langle V, \in \rangle \cong \langle V^{V[\![X]\!]}, \bar{\in} \rangle$.

Interpreting U in V

Corollary.

The following theories are mutually interpretable.

- ZF
- $\mathrm{ZFCU_R} + \mathscr{A} \sim \omega$
- ...
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Proof.

$V[\![\omega]\!]$ has ω-many urelements, and $V[\![Ord]\!]$ has unboundedly many. □

Folklore

$\mathrm{ZFCU_R} \nvdash \mathrm{RP}^-$.

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Proof.

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Remark. This also shows that $ZFCU_R$ cannot prove the Collection Principle, i.e.,

$$\forall x \in w \exists y \varphi(x, y, p) \rightarrow \exists v \forall x \in w \exists y \in v\ \varphi(x, y, p).$$

Question

When will first-order reflection hold?

Plenitude and Tail

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Definition

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(Tail) Every set of urelements has a tail.

A urelement-characterization of RP

Theorem

Over $ZFCU_R$, *the following are equivalent.*

- *RP*
- RP^-
- *Collection*
- *Plenitude* $\vee$ *Tail*

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This provides a characterization of first-order reflection in terms of urelements.

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Theorem

- $ZFU_R + Plenitude \nvdash RP$ (in fact, Collection);
- $ZFU_R + RP \nvdash (Plenitude \vee Tail)$.

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Theorem

- ZFU_R + *Plenitude* $\nvdash$ *RP* (in fact, *Collection*);
- ZFU_R + *RP* $\nvdash$ (*Plenitude* $\vee$ *Tail*).

Open Questions

- ZFU_R + *Collection* $\vdash$ RP^-?
- ZFU_R + RP^- $\vdash$ *RP*?
- ZFU_R + RP^- $\vdash$ *Collection*?
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Urelement class theory

The language of *urelement class theory* is two-sorted: the first-order variables $w, x, y, z, \ldots$ quantify over sets and urelements, and the second-order variables $X, Y, R, F, \ldots$ quantify over classes.

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(Collection) $\forall x \in w\, \exists y R(x, y) \rightarrow \exists v \forall x \in w \exists y \in v\; R(x, y)$.

(RP) For every X_1, ..., X_n, there is a transitive set t such that for every $x_1, \ldots, x_m \in t$,

$$\varphi(X_1, \ldots, X_n, x_1, \ldots, x_m) \leftrightarrow \varphi^t(X_1 \cap t, \ldots, X_n \cap t, x_1, \ldots, x_m),$$

where φ contains only first-order quantifiers.

Class theories with urelements

Definition

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GBCU= GBU_R + Global Well-Ordering (GWO)
KMCU = KMU_R + GWO.

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(Limitation of Size) All proper classes are equinumerous.

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Theorem (Felgner)

Over KMcU_R,

- *Global Choice* $\not\rightarrow$ *Global Well-Ordering;*
- *Global Well-Ordering* $\not\rightarrow$ *Limitation of Size.*

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Open Question

KMcU_R + Collection + Global Choice $\vdash$ RP?

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- Over $ZFCU_R$, RP and RP^- hold exactly when the urelements are arranged in a certain way (i.e., Plenitude $\vee$ Tail), and they follow from Collection.
- The urelement-characterization doesn't hold in either ZFU_R or $KMcU_R$; first-order reflection does not follow from Collection in $KMcU_R$ (conjecture: same in ZFU_R) and behaves much more like a choice principle.
- With absolute generality, it seems that first-order reflection is no longer part of the iterative conception of set.

RP_2 with Urelements

Recall Bernays' second-order reflection principle.

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where φ can be any formula in the language of class theory.

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Question

Can urelements affect the strength of RP_2?

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Question

Is $V < \mathscr{A}$ consistent with RP_2?

The $U_\kappa(A)$-hierarchy

Definition

Let κ be an infinite cardinal and $A \subseteq \mathscr{A}$.

$$U_\kappa(A) = \bigcup_{B \in P_\kappa(A)} V_\kappa(B),$$

where $P_\kappa(A) = \{x \subseteq A : x < \kappa\}$.

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Zermelo's Quasi-Categoricity Theorem.

A full second-order model $\mathscr{M}$ satisfies ZFC_2 iff $\mathscr{M}$ is isomorphic to some V_κ, where κ is inaccessible.

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$U_\kappa(A)$ is a natural generalization of V_κ in the context of urelement set theory.

Theorem

The following are equivalent.

- $\langle M, P(M)\rangle$ *is a transitive model of KMCU.*
- $M = U_\kappa(A)$ *for some inaccessible cardinal* κ *and* $A \subseteq \mathscr{A}$.

Moreover, $U_\kappa(A) \models V < \mathscr{A}$ *if* $\kappa < A$.

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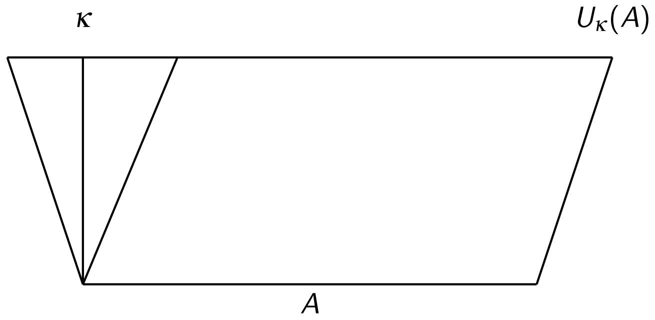

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We can construct an elementary embedding $j : U \to M$ with $crit(j) = \kappa$ such that

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Then $U_\kappa(A) \in M$, and by the elementarity of j, $U_\kappa(A) \models RP_2 + V < \mathscr{A}$. □

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Question

What is the strength of $RP_2 + V < \mathscr{A}$?

Definition (Hamkins, Y.)

The Abundant Atom Axiom (AAA) $=_{\text{df}}$

- $V < \mathscr{A}$;
- for every small class B (i.e, $B < \mathscr{A}$) there is a small $D \subseteq I \times B$ such that every subclass of B is D_i for some $i \in I$ ("$\mathscr{A}$ **strong limit**");
- if I is small and $D \subseteq I \times B$ is such that D_i is small for each $i \in \mathscr{I}$, then D itself is small ("$\mathscr{A}$ **regular**").

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Proposition (Hamkins, Y.)

- If $\kappa < \kappa'$ are both inaccessible and $|A| = \kappa'$, then $U_\kappa(A) \models$ AAA;
- if κ is λ-supercompact for some inaccessible $\lambda > \kappa$, then there is a model of $U_\kappa(A) \models RP_2 +$ AAA.

Theorem (Hamkins, Y.)

KMCU+ RP_2 + AAA interprets KM + a supercompact cardinal.

Proof sketch.

Let U be a model of KMCU + AAA + RP_2. We can then carry out the **unrolling construction** (due to Marek and Mostowski).

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A membership code is an extensional and well-founded class graph with a unique maximal code. Treat equivalence classes of isomorphic membership codes as sets. Define $E \varepsilon F$ as "E is isomorphic to F restricted to an immediate descendant of its maximal node" . This unrolls a model $\langle W, \varepsilon \rangle$ of ZFC^-.

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Ord^U will code an inaccessible cardinal κ in W. The shortest well-ordering of $\mathscr{A}$ will code a cardinal λ above κ. The abundance of $\mathscr{A}$ implies that λ is inaccessible.

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Let $\overline{V} = V_\lambda^W$, which is a model of KM with an inaccessible cardinal κ.

Proof sketch (continued).

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A cardinal θ is **second-order reflective**, if every second-order sentence φ true in some structure M (of any size) with $\theta \subseteq M$ in a language of size less than θ is also true in a first-order elementary substructure $m \prec M$ of size less than θ and with $m \cap \theta \in \theta$.

Theorem (Hamkins, Y.)

θ is second-order reflective if and only if θ is supercompact. □

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RP_2 + AAA in U implies that κ is second-order reflective in $\bar{V}$. □

Corollary

$RP_2 + AAA \vdash V \neq L$ (and more).

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Proof.

V (of U) is isomorphic to $V_\kappa^{\overline{V}}$ by $x \mapsto \langle TC\{x\}, \in \rangle$. In particular, every $y \in V_\kappa^{\overline{V}}$ has a set code, which is isomorphic to the transitive closure of some pure set in U.

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Since κ is supercompact in $\overline{V}$, it follows that there are proper-class many measurable cardinals (and more) in V. □

Philosophical remarks

- Quantifying over everything can indeed affect second-order reflection given enough urelements. This can be seen as another way of strengthening reflection principles in addition to several attempts in the literature.

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- Many (including Gödel) think that Limitation of Size is a **maximality principle**. But Limitation of Size is rather a restrictive axiom with absolute generality: under RP_2, LS holds iff $\mathscr{A} \leq V$. Therefore, it is $\neg$LS, rather than LS, that maximizes the universe.

Thank You!