# Reflective Mereology

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## 1 Set Theory and Mereology

### 1.1 Absolute generality

We may hope to quantify over *everything* in both mereology and set theory. This will provide a natural reading of certain principles in mereology and set theory. E.g.,

> (Transitivity) For *every* $x$, $y$ and $z$, if $x$ is a part of $y$ and $y$ is a part of $z$, then $x$ is a part of $z$

> (Pairing) For *every* $x$ and $y$, there is a set of them.

**Question:** Can mereology sit well with set theory when both quantify over *everything*?

### 1.2 Formal setting

The language of plural quantification $\mathscr{L}^{\propto}$ (monadic second-order logic). $xx, yy, zz...$ stand for pluralities; $x \propto xx$ abbreviates $x$ is among $xx$.

> (*Plural Comprehension*) $\exists yy \forall z (z \propto yy \leftrightarrow \varphi(z))$.

The plural language of mereology and set theory $\mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$ contains three non-logical first-order predicates $<$, $\in$, and Ur for proper parthood, membership, and urelementhood.

The axioms of Classical Mereology ($\mathrm{CM}_2$):

> (Transitivity) $<$ is transitive.

> (Asymmetry) $<$ is asymmetric.

(WeakSup) If $x < y$, then $y$ has a proper part that does not overlap $x$.

(Unrestricted Fusion) Every non-empty *xx* have a fusion. [1]

The appeal of Unrestricted Fusion: fusions are ontologically innocent. $\mathrm{CM}_2$ proves

(Fusion Uniqueness) Every non-empty plurality have a unique fusion.

$\mathrm{ZFCU}_2$ is second-order ZFC with urelements formulated in $\mathscr{L}^{\infty}_{<,\in,\mathrm{Ur}}$.

### 1.3 Uzquiano's cardinality problem

CM + ZFCU already also yields unnatural consequences, e.g., by Unrestricted Fusion, there is a $u$ that fuses everything but then $\{u\}$ must be a proper part of $u$...

To make it worse, consider the following axioms.

(Limitation of Size) *xx* form a set iff there is no surjective map from *xx* onto everything.

(Atomicity) Everything is a fusion of some mereological atoms.

**Theorem 1.1** (Uzquiano)**.** The theory $\mathrm{CM}_2$ + $\mathrm{ZFCU}_2$ + LS + Atomicity is inconsistent.

*Proof.* By a Cantorian argument, we can show that under $\mathrm{CM}_2$, there is no surjective map from all mereological atoms to everything, so the atoms form a set by LS. But by Fusion Uniqueness a set of atoms can only generate a set of fusions. It then follows from Atomicity that there is a set of everything—contradiction. $\square$

Note: full Atomicity is not needed. The argument works as long as the atomless gunks form a set.

Some possible ways out:

- Give up absolute generality.
- Introduce proper classes (Lewis).
- Reject Limitation of Size.
- ...

[1] $x$ is a party of $y$ ($x \leq y$) iff $x < y \vee x = y$. $x$ overlaps $y$ iff they have a common part. $x$ is a fusion of $yy$ iff every $y$ among $yy$ is a part of $x$ and every part of $x$ overlaps some $y$ among $yy$.

- Find another mereology.

A common diagnosis: things fuse too liberally in CM. Some restriction is needed. For example, it might be nice to have

(Fusion $\leftrightarrow$ Set) $xx$ fuse iff $xx$ form a set.

**Question:** Can there will be an alternative mereology that is consistent with $\mathrm{ZFCU}_2$ + LS + Atomicity?

# 2 Reflective Mereology

## 2.1 Set-theoretic reflection

**Definition 2.1.** Given a set $s$ and some set-theoretic assertion $\varphi(xx)$ with possibly some plural parameters, $\varphi(xx)^{\in s}$ is the result of restricting all the quantifiers of $\varphi$ to the *members* of $s$ and replacing every occurrence of $xx$ with $xx \cap s$.

Reflection principles in set theory amounts to the indescribability of $V$: any true statement about $V$ is already true in some transitive set. Bernays' second-order reflection:

($\mathrm{RP}_2$) $\varphi(xx) \to \exists s(s \text{ is a transitive set} \wedge \varphi(xx)^{\in s})$.

Why $\mathrm{RP}_2$ in set theory?

- It provides a unified justification for many axioms of $\mathrm{ZFC}_2$.
- It produces a fair amount of large cardinals.

## 2.2 Reflective mereology

If $V$ is indescribable, so should be the whole reality. Our mereology should reflect this fact: **whatever is true is already true within some object.**

**Definition 2.2.** Given an object $t$ and some assertion $\varphi(xx) \in \mathscr{L}^{\propto}_{<,\in,\mathrm{Ur}}$ with possibly some plural parameters, $\varphi(xx)^{<t}$ is the result of restricting all the quantifiers of $\varphi$ to the **proper parts** of $t$ and replacing every occurrence of $xx$ with the proper parts of $t$ among $xx$.

($\mathrm{MRP}_2$) $\varphi(xx) \to \exists t \varphi(xx)^{<t}$.

I shall also assume a weaker version of Unrestricted Fusion.

(M-Separation) If $xx$ are parts of some $y$, then $xx$ have a fusion.

The idea behind M-Separation: when it is safe to fuse, fuse as much as possible.

**Definition 2.3.** The axioms of *Reflective Mereology* ($RM_2$) consist of Transitivity, Asymmetry, M-Separation and all instances of $MRP_2$.

**Lemma 2.1.** Assume $RM_2$. Then,

(1) everything is a proper part of something;

(2) nothing contains everything as a part;

(3) every finite plurality have a fusion. □

Moreover, $RM_2$ proves that things fuse when they are not too many

**Theorem 2.2** (Mereological Replacement)**.** Assume $RM_2$. If $xx$ fuse and there is a surjective map from $xx$ onto $yy$, then $yy$ fuse. □

## 2.3 RM $\vdash \neg$ Weak Supplementation

**Lemma 2.3.** Assume $RM_2$ + WeakSup. For every $x$ and $y$,

(i) if every part of $y$ overlaps $x$, then $y$ is a part of $x$;

(ii) if $y$ is a part of $x$ and everything disjoint from $y$ is also a part of $x$, then everything is a part of $x$. □

**Lemma 2.4.** Assume $RM_2$ + WeakSup. For every $x$, the objects that are disjoint from $x$ have a fusion.

*Proof.* Fix an $a$ and let $dd$ be the plurality of all things disjoint from $a$. Suppose that $dd$ don't have a fusion. Fix a $u$ that is disjoint from $a$, which exists by WeakSup and 2.1. Then by MRP, there is a $t$ with $a, u < t$ such that

$$\forall z < t \neg Fu(z, dd)^{<t}.$$

Thus, for all $z < t$, $\neg Fu(z, dd)^{<t}$. But consider $dd_t$ which are things among $dd$ that are proper parts of $t$. By M-Separation, $dd_t$ have a fusion $z$, so $Fu(z, dd_t)$. $z$ is a part of $t$ by Lemma 2.3 and indeed a proper part of $t$ because $a < t$. So it follows from $Fu(z, dd_t)$ that $Fu(z, dd)^{<t}$—contradiction. □

**Theorem 2.5.** $RM_2$ + Weak Supplementation is inconsistent.

*Proof.* Fix some $a$. By the last lemma, $dd_a$ have fusion $b$. By finitary fusion, some $u$ fuses $a$ and $b$. Then by Lemma 2.3 (ii), it follows that everything is a part of $u$, which contradicts 2.1 (ii). □

**Corollary 2.5.1.** Assume $RM_2$. Then some plurality have more than one fusion.

*Proof.* By a standard argument as in CM but only using Finitary Fusion, one can show that Weak Supplementation is equivalent to Fusion Uniqueness. □

# 3 A Natural Model of $RM_2$ + $ZFCU_2$

Let $U$ be a full second-order model of $ZFCU_2$ + Limitation of Size + $RP_2$ with a set of urelements of size $2^\kappa$ for some infinite cardinal $\kappa$. In $U$, we first define a parthood notion on the set of urelements Ur by fixing a bijection $f$ from Ur to $P(\kappa)\setminus\{\emptyset\}$.

**Definition 3.1.** For every $x, y \in U$

(i) $x \sqsubset y$ ($x$ is a proper Ur-*part* of $y$) iff $x$ and $y$ are urelements with $f(x) \subsetneq f(y)$;

(ii) $x < y$ ($x$ is a proper part of $y$) iff either $x \sqsubset y$, or $x \in TC(y)$, or $x \sqsubset z$ for some $z \in TC(y)$.

Let $\mathscr{U} = \langle U, P(U), \in^U, \mathrm{Ur}, < \rangle$ be the corresponding model for the language $\mathscr{L}^{\infty}_{<,\in,\mathrm{Ur}}$.

**Theorem 3.1.** $\mathscr{U} \models$ Atomicity $\wedge$ $RM_2$. Moreover, in $\mathscr{U}$ Classical Merelogy holds the urelements, and Fusion $\leftrightarrow$ Set holds.

*Proof. Atomicity*. Let $x$ be a set. Then either $\emptyset$ or some atomic urelement will be an atomic part of $x$.

*M-Separation*. Let $xx$ be parts of some $y$. Then $xx$ are included by $TC(\{y\}) \cup \mathrm{Ur}$, which is a set. It is easy to check that the set of $xx$ will be a fusion of $xx$.

*$MRP_2$* (Sketch). The key observation is that since $<$ is a definable relation in Ur, for every transitive set $t$ that is tall enough and contains all urelements, we shall have

$$\varphi^{\in t} \leftrightarrow \varphi^{<t}.$$

So if $\varphi$ holds, we can find such $t$ by using $RP_2$ to reflect

$$\varphi \wedge ZFCU_2 \wedge \exists y (y = \mathrm{Ur}).$$

□